AERODYNAMICS PROBLEM SET 1
SOLUTIONS

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Problem 1

Eq.2.74 of the textbook gives an expression for the drag of a body in terms of the freestream velocity, and the downstream velocity and density distribution.

$$D' = \int_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \rho_2 u_2 (u_1 - u_2) dy \tag{1}$$

We are given the downstream velocity distribution in terms of the freestream velocity as the following equation.

$$U = U_\infty \left[1 - 0.83 \cos^2 \left(\frac{\pi y}{b} \right) \right] \tag{2}$$

We will assume incompressible flow so that ρ is constant throughout the flow. We can then integrate the right hand side of Eq. 1 to obtain the drag per unit span.

$$D' = \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho \left[U_\infty \left(1 - 0.83 \cos^2 \left(\frac{\pi y}{b} \right) \right) \right] \left[U_\infty \left(0.83 \cos^2 \left(\frac{\pi y}{b} \right) \right) \right] dy \tag{3}$$

or

$$D' = 0.83 \rho U_\infty^2 \left[\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos^2 \left(\frac{\pi y}{b} \right) dy - 0.83 \int_{-\frac{b}{2}}^{\frac{b}{2}} \cos^4 \left(\frac{\pi y}{b} \right) dy \right] \tag{4}$$

Integrating gives

$$D' = 0.83 \rho U_\infty^2 \left[\frac{y}{2} + \frac{\sin(\frac{2\pi y}{b})}{\frac{4\pi}{b}} - 0.83 \left(\frac{3y}{8} + \frac{\sin(\frac{2\pi y}{b})}{\frac{4\pi}{b}} + \frac{\sin(\frac{4\pi y}{b})}{\frac{32\pi}{b}} \right) \right]_{-\frac{b}{2}}^{\frac{b}{2}} \tag{5}$$

substituting values for y gives

$$D' = 0.83\rho U_\infty^2 \frac{b}{2} - 0.83^2 \rho U_\infty^2 \frac{3b}{4} \tag{6}$$

for unit span

$$C_d = \frac{D'}{\frac{1}{2}\rho U_\infty^2 c} \tag{7}$$

where c is the chord length. Since b=0.1c, the coefficient of drag is given by

$$C_d = \left[0.83\rho U_\infty^2 \frac{b}{2} - 0.83^2 \rho U_\infty^2 \frac{3b}{8}\right]\left[\frac{2}{\rho U_\infty^2 c}\right] \tag{8}$$

or

$$C_d = (0.83)(0.1) - (0.83^2)(\frac{3}{4})(0.1) \tag{9}$$

therfore $C_d = 0.0313$

Problem 2

The energy equation in 2-dimensions is

$$\rho u \frac{\partial\left(e + \frac{\mathbf{V^2}}{2}\right)}{\partial x} + \rho v \frac{\partial\left(e + \frac{\mathbf{V^2}}{2}\right)}{\partial y} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$
$$-\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} + \frac{\partial}{\partial x}\left[\mu v\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu u\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

where $\mathbf{V}^2 = u^2 + v^2$
The x component of the momentum equation in 2-dimensions is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

multiply the momentum equation by u to get

$$\rho u^2 \frac{\partial u}{\partial x} + \rho uv \frac{\partial u}{\partial y} = -u\frac{\partial p}{\partial x} + u\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

rearrange using the product rule and chain rule to get

$$\rho u \frac{\partial \frac{u^2}{2}}{\partial x} + \rho v \frac{\partial \frac{u^2}{2}}{\partial y} = -\frac{\partial(up)}{\partial x} + p\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left[\mu u\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] - \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y}$$

similarly, for the momentum equation in the y direction

$$\rho u \frac{\partial \frac{v^2}{2}}{\partial x} + \rho v \frac{\partial \frac{v^2}{2}}{\partial y} = -\frac{\partial(vp)}{\partial y} + p\frac{\partial v}{\partial y} + \frac{\partial}{\partial x}\left[\mu v\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] - \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x}$$

Now, subtracting the above equations for momentum from the energy equation gives

$$\begin{aligned}\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y} &= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-p\frac{\partial u}{\partial x}-p\frac{\partial v}{\partial y}\\ &\quad+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y}+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x}\end{aligned}$$

rearrange the above equation to get

$$\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

Now, substitute $e=h-\frac{p}{\rho}$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-\rho u\frac{\partial\left(\frac{p}{\rho}\right)}{\partial x}-\rho v\frac{\partial\left(\frac{p}{\rho}\right)}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

Note that by the product rule

$$\begin{aligned}\rho u\frac{\partial\left(\frac{p}{\rho}\right)}{\partial x} &= \rho up\frac{\partial\left(\frac{1}{\rho}\right)}{\partial x}+u\frac{\partial p}{\partial x}\\ \rho v\frac{\partial\left(\frac{p}{\rho}\right)}{\partial y} &= \rho vp\frac{\partial\left(\frac{1}{\rho}\right)}{\partial y}+v\frac{\partial p}{\partial y}\end{aligned}$$

And with the chain rule,

$$\begin{aligned}\rho up\frac{\partial\left(\frac{1}{\rho}\right)}{\partial x} &= -\frac{up}{\rho}\frac{\partial \rho}{\partial x}\\ \rho vp\frac{\partial\left(\frac{1}{\rho}\right)}{\partial y} &= -\frac{vp}{\rho}\frac{\partial \rho}{\partial y}\end{aligned}$$

So the equation above can be rewritten as

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-u\frac{\partial p}{\partial x}-v\frac{\partial p}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{p}{\rho}\left(u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial v}{\partial y}\right)+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

Recall the continuity equation

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial v}{\partial y}=0$$

So finally, with all the changes, the energy equation becomes

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-u\frac{\partial p}{\partial x}-v\frac{\partial p}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

In non-dimensional form,

$$\frac{c_p\rho_\infty v_\infty T_\infty}{c}\left[\rho' u'\frac{\partial h'}{\partial x'}+\rho' v'\frac{\partial h'}{\partial y'}\right]-\frac{v_\infty p_\infty}{c}\left[u'\frac{\partial p'}{\partial x'}-v'\frac{\partial p'}{\partial y'}\right] = \frac{k_\infty T_\infty}{c^2}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right] + \frac{\mu_\infty v_\infty^2}{c^2}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)^2\right]$$

Or

$$\rho' u'\frac{\partial h'}{\partial x'}+\rho' v'\frac{\partial h'}{\partial y'}-\frac{p_\infty}{\rho_\infty c_p T_\infty}\left[u'\frac{\partial p'}{\partial x'}-v'\frac{\partial p'}{\partial y'}\right] = \frac{k_\infty}{c\rho_\infty v_\infty c_p}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right] + \frac{\mu_\infty v_\infty}{c\rho_\infty c_p T_\infty}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)^2\right]$$

Now,

$$\frac{p_\infty}{\rho_\infty c_p T_\infty} = \frac{\gamma-1}{\gamma}$$
$$\frac{k_\infty}{c\rho_\infty v_\infty c_p} = \frac{1}{Pr_\infty Re_\infty}$$
$$\frac{\mu_\infty v_\infty}{c\rho_\infty c_p T_\infty} = \frac{M_\infty^2}{Re_\infty}$$

Assuming $\frac{1}{Re} \propto O(\delta^2)$, then in order of magnitude form, the energy equation is

$$O(1)+O(1)-O(1)-0(\delta^2) = O(\delta^2)\left[O(1)+O(\frac{1}{\delta^2})\right]-O(\delta^2)\left[O(1)+O(\frac{1}{\delta})+O(\frac{1}{\delta^2})\right]$$

Keeping only the terms of $O(1)$, and returning to dimensional form,

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-u\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

as required

Problem 3

Most people got this question right so I'll just give the answers
$Re = 9.6x10^6$, $\mu = 1.7894x10^{-5}$
From the graph for an angle of attack of 5°, $c_l = 0.73$, $c_m = -0.05$

$$L' = q_\infty c c_l = 4385 N/m$$
$$M' = q_\infty c^2 c_{m_{\frac{c}{4}}} = -600 Nm/m$$

The speed of sound is given by $a = \sqrt{\gamma RT}$, so $M = 0.2$

Problem 4

The camber of the airfoil is described by the equation

$$\left(x-\frac{c}{2}\right)^2+\left(z+\frac{c}{8k}-\frac{kc}{2}\right)^2=\left(\frac{c}{8k}+\frac{kc}{2}\right)^2$$

First we need to find the derivative of z wrt x

$$\frac{dz}{dx}=\frac{-\left(x-\frac{c}{2}\right)}{\sqrt{\left(\frac{c}{8k}+\frac{kc}{2}\right)^2-\left(x-\frac{c}{2}\right)^2}}$$

Applying the transformation $x=\frac{c}{2}(1-\cos\theta)$

$$\begin{aligned}\frac{dz}{dx} &= \frac{\frac{c}{2}\cos\theta}{\sqrt{\left(\frac{c}{8k}+\frac{kc}{2}\right)^2-\left(\frac{c}{2}\right)^2\cos^2\theta}}\\ &= \frac{\cos\theta}{\sqrt{\left(\frac{1}{4k}+k\right)^2-\cos^2\theta}}\\ &= \frac{\cos\theta}{\sqrt{\frac{1}{16k^2}+\frac{1}{2}+k^2-\cos^2\theta}}\end{aligned}$$

Notice that in the denominator the term $\frac{1}{16k^2}$ is far greater than the other terms under the assumption $k \ll 1$. So the derivative reduces to

$$\frac{dz}{dx}=4k\cos\theta$$

Now we can determine $A_0, A_1, A_2, ...$
For A_0

$$\begin{aligned}A_0 &= \alpha-\frac{1}{\pi}\int_0^\pi\frac{dz}{dx}d\theta\\ &= \alpha-\frac{1}{\pi}\int_0^\pi 4k\cos\theta d\theta\\ &= \alpha-\frac{4k}{\pi}\sin\theta\Big|_0^\pi\\ &= \alpha\end{aligned}$$

For A_n

$$A_n = \frac{2}{\pi}\int_0^\pi\frac{dz}{dx}\cos n\theta d\theta$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\pi} 4k\cos\theta\cos n\theta d\theta \\
&= \frac{4k}{\pi}\sin\theta\Big|_0^{\pi}
\end{aligned}
$$

$$
A_n = \begin{cases} 4k & n = 0 \\ 0 & n > 1 \end{cases}
$$

now we can find an expression for γ

$$
\begin{aligned}
\gamma(\theta) &= 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n\sin n\theta\right) \\
\gamma(\theta) &= 2V_\infty\left(\alpha\frac{1+\cos\theta}{\sin\theta} + 4k\sin\theta\right)
\end{aligned}
$$

For the angle of zero lift,

$$
\begin{aligned}
\alpha_{L=0} &= -\frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}(\cos\theta - 1)d\theta \\
&= -\frac{4k}{\pi}\int_0^{\pi}(\cos^2\theta - \cos\theta)d\theta \\
&= -2k
\end{aligned}
$$

For the moment coefficient about the aerodynamic centre

$$
\begin{aligned}
c_{m_{\frac{c}{4}}} &= \frac{\pi}{4}(A_2 - A_1) \\
&= \frac{\pi}{4}(0 - 4k) \\
&= -k\pi
\end{aligned}
$$

Problem 5

Let the equation for the camber line be

$$
z = ax^3 + bx^2 + cx + d
$$

with $0 \leq x \leq 1$
The airfoil has the following properties:
1. $z(0) = 0$
2. $z(1) = 0$
3. $c_{m_{\frac{c}{4}}} = 0$
4. at $\frac{dz}{dx} = 0$, $z = \delta$

The first point means that d=0, so the equation for the camber can be simplified to $z = ax^3 + bx^2 + cx$
now,

$$c_{m_{\frac{c}{4}}} = \frac{\pi}{4}(A_2 - A_1) = 0$$

so $A_2 - A_1 = 0$, where

$$A_2 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos 2\theta d\theta$$

and

$$A_1 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos \theta d\theta$$

differentiating z wrt x

$$\frac{dz}{dx} = 3ax^2 + 2bx + c$$

Applying the transformation $x = \frac{1}{2}(1 - \cos\theta)$, gives us

$$\begin{aligned}\frac{dz}{dx} &= 3a\left(\tfrac{1}{2}\left(1-\cos\theta\right)\right)^2 + 2b\left(\tfrac{1}{2}\left(1-\cos\theta\right)\right) + c \\ &= \tfrac{3a}{4}\cos^2\theta - \left(\tfrac{3a}{2} + b\right)\cos\theta + \tfrac{3a}{4} + b + c\end{aligned}$$

Now,

$$\begin{aligned}A_2 - A_1 &= \tfrac{2}{\pi}\int_0^\pi \left(\tfrac{3a}{4}\cos^2\theta - \left(\tfrac{3a}{2}+b\right)\cos\theta + \tfrac{3a}{4} + b + c\right)\left(\cos 2\theta - \cos\theta\right) d\theta \\ &= \tfrac{2}{\pi}\int_0^\pi \tfrac{3a}{2}\cos^4\theta - \left(\tfrac{15a}{4} + b\right)\cos^3\theta + \left(\tfrac{9a}{4} + 3b + 2c\right)\cos^2\theta \\ &\quad - \left(c - \tfrac{3a}{4}\right)\cos\theta - \left(\tfrac{3a}{4} + b + c\right) d\theta\end{aligned}$$

Integrating and substituting the values for θ gives

$$A_2 - A_1 = \frac{15a}{8} + b$$

For the coefficient of moment to be zero we must have $A_2 - A_1 = 0$ which means $\frac{15a}{8} + b = 0$ or $b = -\frac{15a}{8}$. So the equation for the camber becomes

$$z = ax^3 - \frac{15a}{8}x^2 + cx$$

Applying the constraint that z(1)=0 gives

$$\begin{gathered}a(1^3) - \tfrac{15a}{8}(1^2) + c(1) = 0 \\ -\tfrac{7}{8}a + c = 0 \\ c = \tfrac{7}{8}a\end{gathered}$$

the equation is now $z = ax^3 - \frac{15}{8}ax^2 + \frac{7}{8}ax$. we know that at $\frac{dz}{dx} = 0$, $z = \delta$

$$\frac{dz}{dx} = 3ax^2 - \frac{15}{4}ax + \frac{7}{8}a = 0$$

$$\begin{aligned} x &= \frac{\frac{15}{4} - \sqrt{\left(\frac{15}{4}\right)^2 - 4(3)\left(\frac{7}{8}\right)}}{6} \\ &= 0.3104 \end{aligned}$$

sub back into the equation for the camber line

$$\begin{aligned} z &= a(0.3104)^3 - \tfrac{15}{8}a(0.3104)^2 + \tfrac{7}{8}a(0.3104) = \delta \\ a &= 8.274\delta \end{aligned}$$

so the equation of the camber line is

$$z = 8.274\delta x^3 - 15.5146\delta x^2 + 7.2402\delta x$$

or if $0 \leq x \leq c$, then

$$z = 8.274\delta\left(\frac{x}{c}\right)^3 - 15.5146\delta\left(\frac{x}{c}\right)^2 + 7.2402\delta\left(\frac{x}{c}\right)$$