ASPECTS OF LINEAR STABILITY ANALYSIS FOR HIGHER-ORDER FINITE-DIFFERENCE METHODS

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Abstract

This paper attempts to clarify some issues in stability analysis which are especially pertinent to higher-order finite-difference approximations of hyperbolic partial differential equations. Examples are given which illustrate the difficulties which can arise in analyzing the stability of higher-order methods, especially those which result in non-normal operator matrices. Stability tests discussed include Fourier analysis, asymptotic stability (often referred to as the matrix method), and ϵ-pseudo-eigenvalues. Many of the criticisms of the asymptotic stability test are addressed. In particular, the following conjecture is presented: the combination of stability by Fourier analysis and asymptotic stability is sufficient for Lax-Richtmyer stability and is necessary and sufficient for P-stability. An example is given for which the ϵ-pseudo-eigenvalues fail to reveal an instability. For the same example, the singular value decomposition is shown to be a useful tool in stability analysis.

Introduction

For some applications of interest in computational fluid dynamics, including direct simulations of transition and turbulence [1], computational acoustics and aeroacoustics [2], and even steady flows [3], higher-order discretizations can be advantageous. One of the primary difficulties encountered in applying higher-order finite-difference methods is the need for numerical boundary schemes (NBS's) which preserve the global order of accuracy of the interior scheme and produce a stable algorithm. Progress has been made by Olsson [4], Carpenter et al. [5, 6], and Zingg and Lomax [7]. In order to develop NBS's, a suitable test for stability is required. There are several such tests, each with advantages and disadvantages.

Lax (or Lax-Richtmyer) stability derives its importance from the Lax equivalence theorem, which states that Lax stability is necessary and sufficient for convergence of a consistent numerical algorithm applied to a well-posed problem. The theory of Gustafsson, Kreiss, and Sundstrom [8] (GKS) provides a sufficient test for Lax stability. However, the GKS stability test can be very difficult to apply to higher-order methods. Furthermore, both Lax and GKS stability permit non-physical growth in time. Hence these definitions are generally supplemented by the requirement of asymptotic (or time) stability, which ensures that the solution transient remains bounded as time goes to infinity on a fixed grid [5, 6].

The energy method overcomes some of these objections [9]. However, it can be difficult to define a suitable modified norm (or "energy"). A further disadvantage is that the energy method only provides a sufficient condition for stability. Nevertheless, the energy method is the most promising for use in developing a systematic approach to the development of stable and accurate NBS's for higher-order methods [4, 6].

Fourier analysis is by far the easiest test to apply and is a necessary condition for GKS stability. However, it provides no information with respect to NBS's. The test for asymptotic stability, sometimes called the matrix method, is also fairly straightforward to apply. It has been subjected to a number of criticisms. Its most significant disadvantage is that it does not provide an algebraic test, which makes it difficult to use in a systematic manner to develop NBS's. A further often-mentioned disadvantage is that asymptotic stability permits arbitrarily large transient growth at finite times for non-normal operators. Thus it is not sufficient for Lax stability.

Reddy and Trefethen [10] use the pseudo-eigenvalue concept to provide a necessary and sufficient condition for stability of non-normal operators. Another fairly straightforward method of assessing stability is to study

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*Associate Professor, http://goldfinger.utias.utoronto.ca/~dwz the behavior of powers of the L_2 norm of the operator matrix. This approach shares the disadvantage of the matrix method in that it does not provide an algebraic test.

As a result of the difficulty in applying GKS analysis and the energy method, the simpler tests are often used. For example, Zingg et al. [11] rely on Fourier analysis and the asymptotic stability test to analyze the stability of their method. They also examine the behavior of powers of the norm of their operator matrix. Rai and Chakravarthy [3] also use the latter approach.

The purpose of this paper is to examine practical aspects of some stability tests. Difficulties with the asymptotic stability test are addressed. In particular, we assert that, when used together with Fourier analysis, asymptotic stability provides a reliable criterion which is straightforward to apply. An example is given for which the pseudospectrum fails to reveal a mild instability. We also show that the singular value decomposition (SVD) is a helpful tool in understanding and assessing stability.

Preliminaries

We consider finite-difference approximations to homogeneous linear scalar partial differential equations (PDE's), which can be written in the form

$$\mathbf{u}^{n+1} = \mathbf{C}\mathbf{u}^n \tag{1}$$

where $\mathbf{u}$ is a vector of length M' containing the solution, $\mathbf{C}$ is an $M' \times M'$ matrix, and the superscript is the time step index. The vector $\mathbf{u}$ contains the solution at all of the time levels required by the differencing scheme. For example, for a two-step time-marching method,

$$\mathbf{u}^n = \left[\begin{array}{c} \mathbf{U}^n \\ \mathbf{U}^{n-1} \end{array} \right] \tag{2}$$

where $\mathbf{U}$ is a vector of length M containing the solution at a specific time level. We will concentrate here on one-step methods and will refer to the eigenvalues of $\mathbf{C}$ as σ eigenvalues.

In many algorithms, spatial differencing is performed first, leading to a system of ordinary differential equations (ODE's) in the form

$$\frac{d\mathbf{U}}{dt} = \tilde{\mathbf{A}}\mathbf{U} \tag{3}$$

where $\tilde{\mathbf{A}}$ is an $M \times M$ matrix. Integration in time is then performed using a time-marching method. If a one-step method is used, then M' is equal to M, and $\mathbf{u}$ is equal to $\mathbf{U}$.

There are several different definitions of stability for finite-difference approximations of PDE's. One of the most important is that of Lax and Richtmyer, which requires that there exist a constant $K \geq 1$ such that, for an arbitrary initial condition $\mathbf{u}^0$,

$$\|\mathbf{u}^n\| \leq K\|\mathbf{u}^0\| \tag{4}$$

for all $n \geq 0$, $0 \leq n\Delta t \leq T$ with T fixed. As the time step Δt is reduced, the mesh spacing must be reduced in an appropriate manner; for example, Δx must be proportional to Δt for hyperbolic problems. A necessary and sufficient condition for Lax-Richtmyer stability is

$$\|\mathbf{C}^n\| \leq K \tag{5}$$

for the same conditions as above. With the inequality

$$\|\mathbf{C}^n\| \leq \|\mathbf{C}\|^n \tag{6}$$

this gives

$$\|\mathbf{C}\| \leq 1 \tag{7}$$

as a sufficient condition for Lax-Richtmyer stability.

An alternative definition of stability is given by GKS stability [8], which provides an algebraic test that is necessary and sufficient. For many applications, a more strict definition of stability is required. Consequently, Beam et al. [15] introduced the concept of P-stability, which requires that a method be both GKS stable and asymptotically stable, which is also the approach taken in [6].

For a fully-discrete finite-difference scheme, asymptotic stability requires that the spectral radius of the operator matrix $\mathbf{C}$ be less than or equal to unity. If the spectral radius is less than unity, this ensures that the solution will eventually decay to zero. It does not prevent the solution from growing arbitrarily large at finite times. Hence the spectral radius condition provides a necessary condition for P-stability but not a sufficient one. The asymptotic stability test can also be applied to the semi-discrete operator $\tilde{\mathbf{A}}$, the condition for stability being that the eigenvalues of $\Delta t\tilde{\mathbf{A}}$ (where Δt is the time step) must lie within the stable region of the time-marching method, which can generally be determined from its characteristic polynomial. We will refer to a semi-discrete scheme as being asymptotically stable if all of the eigenvalues of $\tilde{\mathbf{A}}$ lie in the left half of the complex plane.

Fourier analysis provides a means to analyze the interior differencing scheme alone, in the absence of boundary schemes or conditions. It determines whether the amplitude of a solution initiated by a spatially periodic function will grow or decay with time. Since stability of the interior scheme is a necessary condition for Lax-Richtmyer, GKS, and P-stability, Fourier analysis should be performed first. For a one-step method, it is equivalent to applying the spectral radius condition for asymptotic stability to the circulant matrix operator arising from the interior scheme with periodic boundary conditions.

Since our primary interest here is in modelling convection and wave propagation phenomena, we will restrict our attention to the linear convection equation, given by

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \tag{8}$$

where $U = U(x, t)$ and a, the phase speed, is a positive real constant. Stability issues related to diffusion terms and approximations to second derivatives in space are not addressed. Consequently, non-normality in the spatial operator matrix arises from numerical dissipation only, not physical dissipation.

Fourier Stability Analysis

Fourier analysis is described in virtually all textbooks on computational fluid dynamics. Therefore we will not present it here. Instead we make a few points regarding higher-order methods and multi-dimensions.

For a semi-discrete operator, it is useful to examine the modified wavenumber, κ^*, obtained by applying the spatial difference operator to a solution in the form $e^{i\kappa x}$, where κ is the wavenumber. For a centered difference scheme, the modified wavenumber is purely real. Similarly, the eigenvalues of the matrix $\tilde{\mathbf{A}}$ resulting from a centered difference approximation to the spatial derivative in the linear convection equation with periodic boundary conditions are purely imaginary. The maximum value of the modified wavenumber is proportional to the maximum imaginary eigenvalue of $\Delta t\tilde{\mathbf{A}}$. The constant of proportionality is the Courant number, $C = a\Delta t/\Delta x$.

Fig. 1 shows the modified wavenumbers for second, fourth, and sixth-order centered difference schemes. (On the figure, Δx refers to the grid spacing.) As the accuracy of the spatial operator is increased, the maximum value of $\kappa^*\Delta x$ is also increased. Therefore, higher-order finite-difference methods generally require reduced Courant numbers for stability. This does not imply a reduced time step compared to the second-order method, however, since higher-order methods allow a larger grid spacing for equivalent accuracy.

In order to assess stability in two dimensions, the following equation can be used:

$$\frac{\partial U}{\partial t} + a\cos\theta\frac{\partial U}{\partial x} + a\sin\theta\frac{\partial U}{\partial y} = 0 \tag{9}$$

This equation governs a simple plane wave convecting a scalar with speed a along a line making an angle θ with respect to the x-axis. With centered-difference schemes, the limiting case for stability occurs when $\theta = 45°$, assuming that the grid is square and the same operator is used for both spatial derivatives. With $\theta = 45°$, the maximum eigenvalue of the spatial operator matrix is $\sqrt{2}$ times the maximum in the one-dimensional case. Therefore, the time step must be reduced by $\sqrt{2}$ in two-dimensional computations. The factor is $\sqrt{3}$ in three dimensions. In many

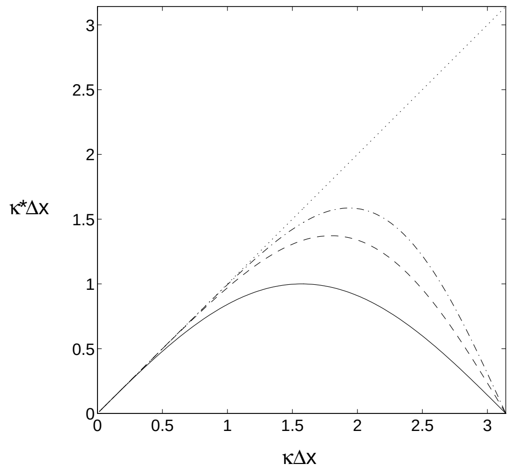

Figure 1: Modified wavenumbers for second-order (——), fourth-order (- - -), sixth-order ($-\cdot-$), and exact ($\cdots$) differencing.

solvers, the Courant number is defined in such a way that this factor is transparent.

Similarly, one can assess stability for finite-volume methods applied to unstructured triangular grids by considering a grid consisting of equilateral triangles. On such a grid, the popular centered finite-volume method used, for example, by Mavriplis [13], is identical to the second-order finite-difference method given by Zingg and Lomax [14]. The modified wavenumber is given by [14]:

$$\begin{aligned} \kappa^*\Delta x \quad = \quad & \frac{2}{3L}\cos\theta\left[\cos\left(\frac{\sqrt{3}\kappa L\sin\theta}{2}\right)\right. \\ & \left.\times\sin\left(\frac{\kappa L\cos\theta}{2}\right) + \sin(\kappa L\cos\theta)\right] \\ & + \frac{2}{\sqrt{3}L}\sin\theta\left[\cos\left(\frac{\kappa L\cos\theta}{2}\right)\right. \\ & \left.\times\sin\left(\frac{\sqrt{3}\kappa L\sin\theta}{2}\right)\right] \end{aligned} \tag{10}$$

where L is the distance between grid nodes. The limiting case for stability is $\theta = 0°$. At this angle, the time step must be reduced by a factor of 1.17 relative to a second-order centered-difference scheme in one dimension. For example, the Courant number limit for such a scheme when used with fourth-order Runge-Kutta time marching is $C = a\Delta t/L \leq 2\sqrt{2}/1.17$. This contrasts with a limit of $2\sqrt{2}$ in the one dimensional case, and 2 in the square grid case with second-order centered differences, Once again, this time step reduction is often hidden in the Courant number definition. One common Courant number definition for unstructured grids uses the height of the triangles as the length scale. For a grid of equilateral triangles, this introduces a factor of $\sqrt{3}/2 = 1.15$, which very nearly cancels the above factor of 1.17.

Asymptotic Stability

In this section, we will consider the asymptotic stability of finite-difference approximations to the linear convection equation on a finite domain, $0 \leq x \leq 1$, with $U(0,t) = 0$ on a mesh consisting of M subintervals of length $\Delta x = 1/M$. After approximating the spatial derivative in eq. 8 by a finite-difference expression, the following system of ODE's is obtained

$$\frac{d\mathbf{U}}{dt} = \tilde{\mathbf{A}}\mathbf{U} \tag{11}$$

where

$$\tilde{\mathbf{A}} = \frac{a}{\Delta x}\mathbf{A} \tag{12}$$

The matrix $\mathbf{A}$ is an $M \times M$ matrix which is generally banded. In the interior of $\mathbf{A}$, the entries are constant along diagonals. When higher-order difference operators are used, NBS's are required in the first and last few rows. Such matrices are called quasi-Toeplitz [15]. We will refer to the eigenvalues of $\mathbf{A}$ as λ eigenvalues

For asymptotic stability, we require the eigenvalues of $\mathbf{A}$ to have nonpositive real parts. Furthermore, Gustaffson [16] has shown that the local accuracy of an NBS should be no more than one order lower than that of the interior scheme in order to preserve the order of accuracy of the global solution. Fig. 2 shows the eigenvalues of $\mathbf{A}$ obtained using sixth-order centered differences in combination with NBS's based on standard fifth-order operators for $M = 100$. The method is clearly asymptotically unstable. The problem is associated with the NBS at the inflow boundary, $x = 0$. Also shown on the figure is the eigenvalue spectrum of the same interior operator with the inflow NBS presented by Jurgens and Zingg [17], which produces asymptotic stability. This fifth-order NBS was derived by enlarging the stencil at the first two grid nodes and using the resulting parameters to provide stability. It is given by

$$(\delta_x U)_1 = \frac{1}{60\Delta x}[-3U_0 - 119U_1 + 255U_2 - 240U_3 + 155U_4 - 57U_5 + 9U_6] \tag{13}$$

$$(\delta_x U)_2 = \frac{1}{60\Delta x}[9U_0 - 66U_1 + 70U_2 - 60U_3 + 75U_4 - 34U_5 + 6U_6] \tag{14}$$

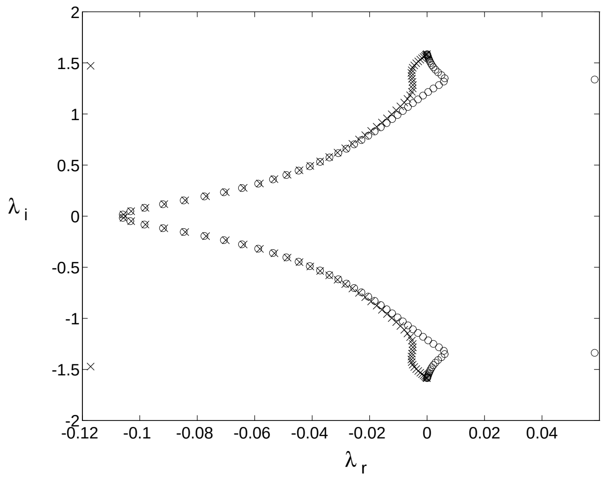

Figure 2: Eigenvalue spectra for sixth-order centered differences with conventional fifth-order NBS (o) and stable fifth-order NBS (x).

Fig. 3 illustrates the effect of increasing M for the spatial operator matrix obtained from sixth-order centered differences coupled with the above NBS at inflow and standard fifth-order operators at outflow. Most of the eigenvalues approach the asymptotic spectrum of the pure Toeplitz matrix which results from Dirichlet boundary conditions if no NBS's are used [15], which is imaginary. The remaining two eigenvalues, which are known as boundary-condition-dependent eigenvalues, are quite insensitive to M. The asymptotic spectrum for Toeplitz matrices can be computed using the algorithm given by Beam and Warming [15]. Thus asymptotic stability can be assessed by computing the asymptotic spectrum and the spectra of a few matrices of moderate size to determine the boundary-condition-dependent eigenvalues and the extent of the deviation from the asymptotic spectrum at finite M.

A difficulty associated with the asymptotic stability test is that numerical methods for computing eigenvalues are unreliable for non-normal matrices of large order [15]. For example, consider the tridiagonal Toeplitz matrix with nonzero entries in each row given by $3/8$, $-1/2$, and $1/8$, which can arise from a spatial discretization of the linear convection-diffusion equation with Dirichlet boundary conditions. Analytically, the eigenvalues of this matrix are all real, with amplitude less than unity. For $M = 100$, a standard eigenvalue routine produces eigenvalues with imaginary parts greater than 0.05, as shown in Fig. 4. The eigenvalues calculated from the transpose are purely real, as desired. Hence it is always a good practice to check the eigenvalues computed from both the matrix and its transpose when dealing with non-normal matri-

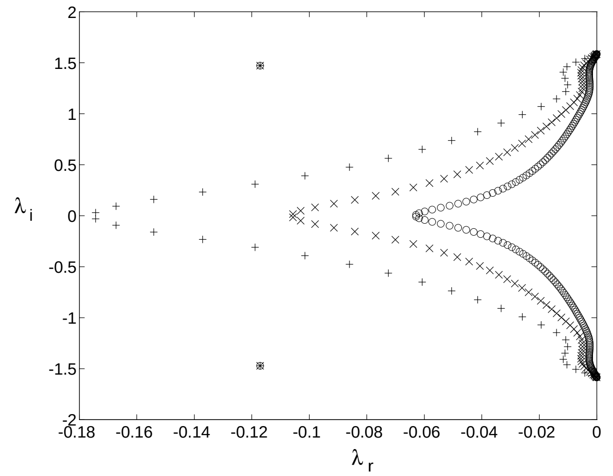

Figure 3: Eigenvalue spectra for sixth-order centered differences with stable NBS, $M = 50$ (+), $M = 100$ (x), and $M = 200$ (o).

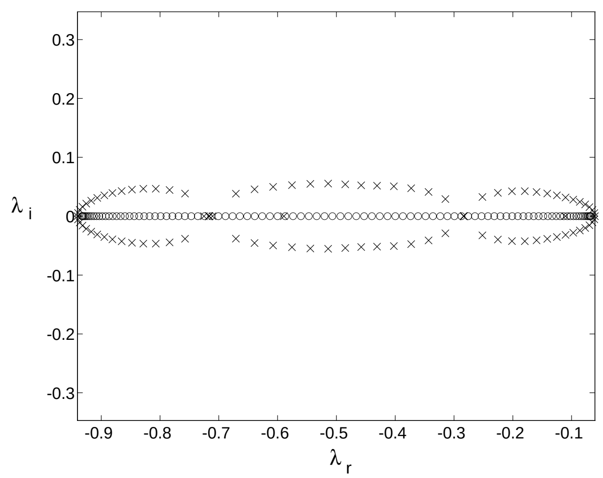

Figure 4: Eigenvalue spectra computed for the Toeplitz matrix B(3/8,-1/2,1/8) with $M = 100$ (x), and its transpose (o).

ces. Furthermore, Beam and Warming [15] have developed a diagonal similarity transformation which dramatically improves the accuracy of eigenvalue computations for Toeplitz and quasi-Toeplitz matrices.

In the late 1970's, some confusion arose with respect to the applicability of the matrix method, primarily in the context of the linear convection-diffusion equation, as summarized by Hirsch [18]. This resulted from attempts to use the matrix method to provide sufficient conditions for stability, which is valid only if the matrix operator is normal. For a normal matrix, the spectral radius is equal to the L_2 norm, and eq. 6 becomes an equality:

$$\|\mathbf{C}^n\| = \|\mathbf{C}\|^n = \rho(\mathbf{C})^n \tag{15}$$

where $\rho(\mathbf{C})$ is the spectral radius of $\mathbf{C}$. Hence asymptotic stability is both necessary and sufficient for P-stability if $\mathbf{C}$ is normal. In the non-normal case, the spectral radius is the lower bound of all norms. Thus asymptotic stability remains necessary but is no longer sufficient.

This can be better understood by considering the numerical group velocity, which is the velocity at which waves transport energy. It is given by

$$a_g = a\frac{dz^*}{dz} \tag{16}$$

where $z^* = \kappa^*\Delta x$ and $z = \kappa\Delta x$. Thus the numerical group velocity is zero for the wavenumber corresponding to the maximum value of $\kappa^*\Delta x$. A wave packet which consists of a modulated function at this wavenumber will not propagate through the grid to the boundary. Consequently, if such a mode is unstable, then the introduction of boundary conditions cannot lead to an asymptotically stable scheme. Conversely, unstable modes with *positive* group velocities can propagate to the outflow boundary and out of the domain. Although such modes grow during the simulation, they eventually decay to zero as they pass through the boundary. If such modes are present, an unstable scheme can be asymptotically stable. Hence asymptotic stability is not a sufficient condition for Lax-Richtmyer, GKS, or P-stability.

We now consider two examples of methods which are unstable by Fourier analysis but asymptotically stable for specific values of Courant number. Both examples are considered further in the next section. The first is the Lax-Wendroff method applied to the linear convection equation with Dirichlet boundary conditions. With these boundary conditions, the eigenvalues of the operator matrix $\mathbf{C}$ can be obtained analytically. Although one might assume that such boundary conditions cause difficulties, very similar behavior is obtained with standard inflow-outflow boundary conditions. Fig. 5 shows the maximum value of $|\sigma|$ obtained from the eigenvalues of $\mathbf{C}$ as well as that obtained from Fourier analysis as a function of the Courant number, for $M = 100$. The condition for stability is $|\sigma| \leq 1$. The eigenvalues of $\mathbf{C}$ indicate that the method is stable up to about $C = 1.15$ (thus it is *asymptotically* stable up to this value), but Fourier analysis gives the correct stability bound, $C = 1$.

For a second example, we consider a finite-difference method which is unstable by Fourier analysis for a range of wavenumbers for which the numerical group velocity is close to the phase velocity. The instability is mild, with a maximum value of $|\sigma|$ of 1.0012 at a Courant number of unity. Since the unstable modes have positive group ve-

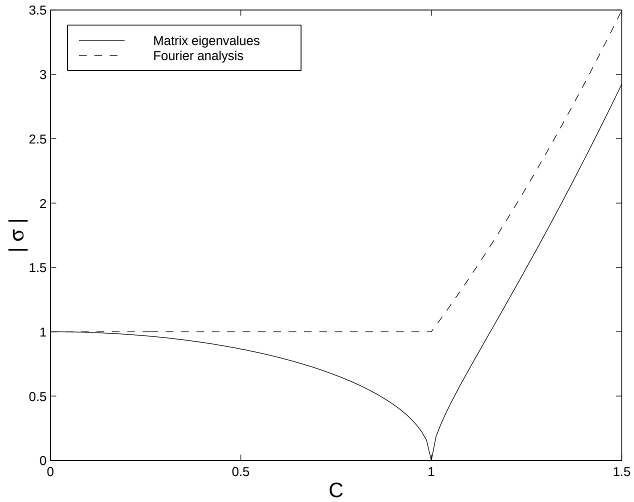

Figure 5: $|\sigma|$ vs. Courant number for the Lax-Wendroff method with Dirichlet and periodic boundary conditions.

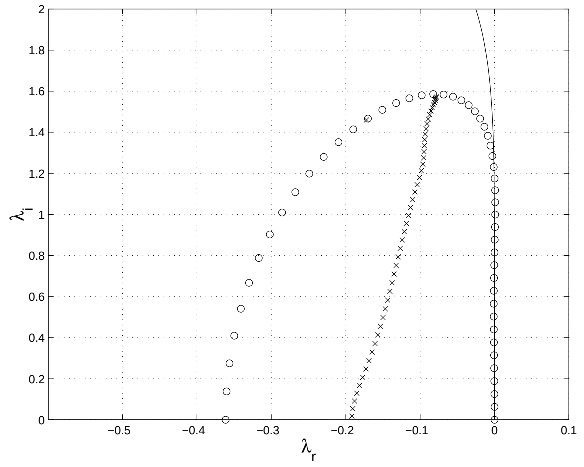

Figure 6: λ eigenvalues with NBS's (x) and periodic boundary conditions (o), and the stability contour of the time-marching method (——) for the unstable scheme.

locities, the scheme is asymptotically stable. The spatial operator is given by sixth-order centered differences plus a symmetric operator given by

$$\begin{aligned}(\delta_x^s U)_j &= \frac{1}{\Delta x}\left[d_3\left(U_{j+3}+U_{j-3}\right)+d_2\left(U_{j+2}+U_{j-2}\right)\right. \\ &\quad \left. +d_1\left(U_{j+1}+U_{j-1}\right)+d_0 U_j\right] \qquad (17)\end{aligned}$$

with $d_0 = 0.1$, $d_1 = -0.0804523$, $d_2 = 0.0403545$, and $d_3 = -0.00990220$. (These coefficients are not recommended for practical use.) When applied to

$$\frac{dU}{dt} = f(U,t) \qquad (18)$$

the six-stage time-marching method is given by [11]:

$$\begin{aligned}U^{(1)}_{n+\alpha_1} &= U_n + \Delta t\alpha_1 f_n \\ U^{(2)}_{n+\alpha_2} &= U_n + \Delta t\alpha_2 f^{(1)}_{n+\alpha_1} \\ U^{(3)}_{n+\alpha_3} &= U_n + \Delta t\alpha_3 f^{(2)}_{n+\alpha_2} \qquad (19) \\ U^{(4)}_{n+\alpha_4} &= U_n + \Delta t\alpha_4 f^{(3)}_{n+\alpha_3} \\ U^{(5)}_{n+\alpha_5} &= U_n + \Delta t\alpha_5 f^{(4)}_{n+\alpha_4} \\ U_{n+1} &= U_n + \Delta t f^{(5)}_{n+\alpha_5}\end{aligned}$$

where $U_n = U(t_n)$, and

$$f^{(k)}_{n+\alpha} = f(U^{(k)}_{n+\alpha}, t_n + \alpha h)$$

with $\alpha_1 = 1/6$, $\alpha_2 = 1/5$, $\alpha_3 = 1/4$, $\alpha_4 = 1/3$, $\alpha_5 = 1/2$.

Fig. 6 shows the stability contour of the time-marching method with a solid line. The eigenvalues of the associated circulant matrix are shown by the symbol "o". Some lie just outside the stable region of the time-marching method. The eigenvalue spectrum of the spatial operator matrix with NBS's and $M = 100$ is shown by the symbol "x", clearly demonstrating the asymptotic stability of the method for this value of M. The NBS at the inflow boundary ($x = 0$) is as given in eqs. 13 and 14. At the outflow boundary ($x = 1$), fifth-order upwind and upwind-biased operators are used. These are given in the Appendix.

Numerous other examples are available which demonstrate that a method which leads to a non-normal operator matrix can be unstable even though it is asymptotically stable. In all of the examples this author has seen, Fourier analysis correctly determines the stability bounds. Along with the above discussion regarding group velocity, this leads to the following conjecture:

Conjecture 1 *The combination of stability by Fourier analysis and asymptotic stability is sufficient for Lax-Richtmyer stability and is necessary and sufficient for P-stability.*

Pseudo-Eigenvalues

When a matrix is non-normal, the eigenvector matrix which diagonalizes it can be very ill-conditioned. In the extreme case of a defective matrix, a full set of linearly independent eigenvectors does not exist, and the matrix cannot be diagonalized. In this case, the solution for an asymptotically stable system includes both polynomial growth and exponential decay. Hence the solution can grow very large before eventually decaying to zero. A non-normal matrix can produce similar behavior even when it

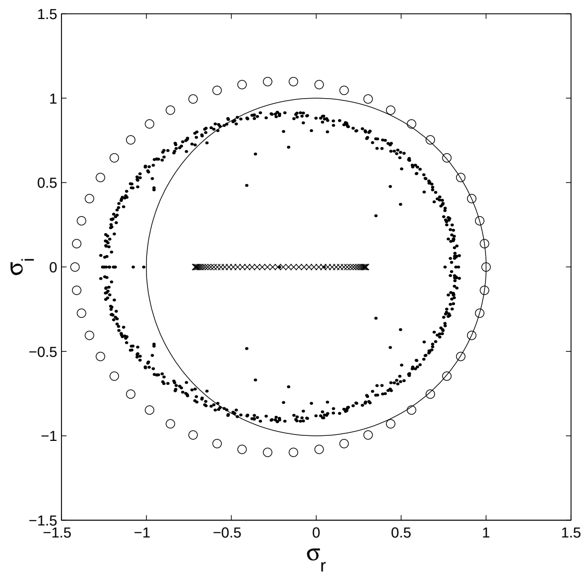

Figure 7: σ eigenvalues for the Lax-Wendroff method at $C = 1.1$ with Dirichlet (x) and periodic boundary conditions (o); ϵ-pseudo-eigenvalues $(\cdot)$ with $\epsilon = 0.001$.

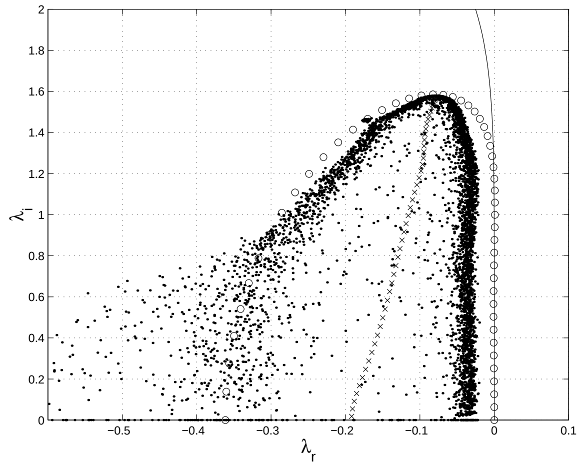

Figure 8: λ eigenvalues with NBS's (x) and periodic boundary conditions (o), ϵ-pseudo-eigenvalues with $\epsilon = 0.01$ $(\cdot)$, and the stability contour of the time-marching method (——) for the unstable scheme.

is nondefective as a result of the lack of orthogonality of its eigenvectors.

Reddy and Trefethen [10] use the concept of ϵ-pseudo-eigenvalues to provide a necessary and sufficient condition for stability for non-normal matrices. The ϵ-pseudo-spectrum of a matrix $\mathbf{A}$ can be computed by calculating the eigenvalues of a number of perturbed matrices $\mathbf{A} + \mathbf{E}$, where each $\mathbf{E}$ is a random complex matrix with norm ϵ. The condition for stability of a semi-discrete approximation is that the ϵ-pseudo-spectrum of the spatial operator matrix lie within ϵ of the stable region of the time-marching method. In the fully-discrete case, the ϵ-pseudo-spectrum of the matrix operator $\mathbf{C}$ must lie within a circle centered on the origin with radius $1 + \epsilon$. For a normal matrix, the ϵ-pseudo-spectrum lies within ϵ of the eigenvalue spectrum. However, for a non-normal matrix the deviation can be large.

Fig. 7 shows the ϵ-pseudo-spectrum for the fully-discrete Lax-Wendroff method with $\epsilon = 0.001$, $C = 1.1$, and $M = 100$. Boundary conditions are Dirichlet. At this Courant number, the method is unstable by Fourier analysis but asymptotically stable, as shown by the eigenvalues displayed in the figure. The pseudo-eigenvalues lie well outside the unit circle, and thus they reveal the instability of the method at this Courant number, as expected.

Fig. 8 shows the ϵ-pseudo-spectrum for our second unstable example with $\epsilon = 0.01$. The pseudo-eigenvalues lie well within the stability contour of the time-marching method. Hence, for this example, the pseudo-eigenvalues give no indication of the instability. Fig. 9 shows the ϵ-pseudo-spectrum with $\epsilon = 0.1$. The pseudo-eigenvalues lie within ϵ of the stability contour, again erroneously suggesting that the scheme is stable. Finally, Fig. 10 shows the pseudo-eigenvalues for the fully-discrete operator matrix, with the same conclusion. Further study is required to understand why the pseudo-eigenvalue approach appears to fail for this example.

Matrix Norms and the SVD

Since uniform boundedness of $\|\mathbf{C}^n\|$ is a necessary and sufficient condition for Lax-Richtmyer stability, studying the behavior of $\|\mathbf{C}^n\|$ for various values of M provides a means of gaining confidence in a method's stability. In this section, we apply this approach to the unstable finite-difference method discussed above. Fig. 11 shows the behavior of $\|\mathbf{C}^n\|$ for three different values of M and a Courant number of unity. It appears that the maximum value of $\|\mathbf{C}^n\|$ does not increase beyond 9.341 (reached at $n = 12$) with increasing M and hence the scheme is stable. A singular value decomposition of $\mathbf{C}^{12}$ reveals that the singular vector associated with the maximum singular value is related to the NBS at the inflow boundary, as shown in Fig. 12. This singular vector, which has unit norm, is a wave packet with a wavenumber which leads to a negative group velocity. This wave packet thus travels towards the inflow boundary and reflects, producing the vector shown in Fig. 13 after 12 time steps, which has a norm of 9.341. It then convects to the outflow boundary and leaves the domain. The growth of this vector obscures the instability resulting from the interior scheme.

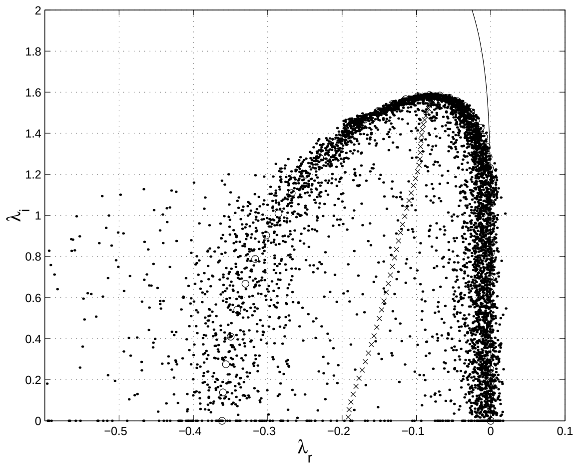

Figure 9: λ eigenvalues with NBS's (x) and periodic boundary conditions (o), ϵ-pseudo-eigenvalues with $\epsilon = 0.1$ ($\cdot$), and the stability contour of the time-marching method (—–) for the unstable scheme.

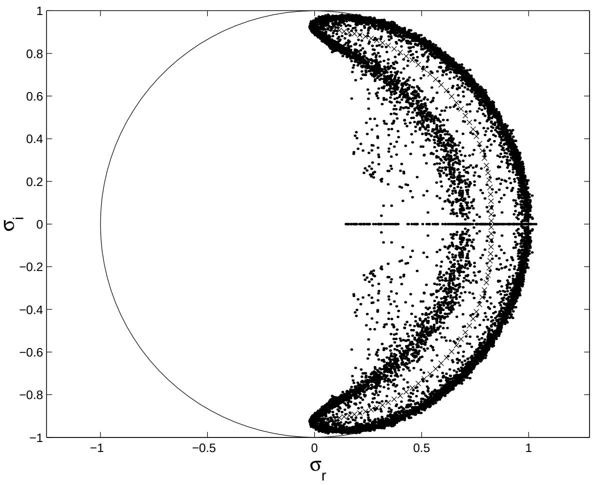

Figure 10: σ eigenvalues with NBS's (x), ϵ-pseudo-eigenvalues with $\epsilon = 0.1$ ($\cdot$), and the stability contour of the time-marching method (—–) for the unstable scheme.

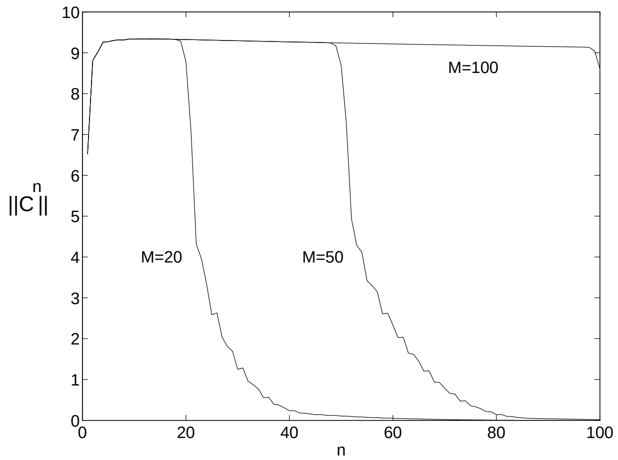

Figure 11: $\|\mathbf{C}^n\|$ for the unstable scheme with the inflow NBS.

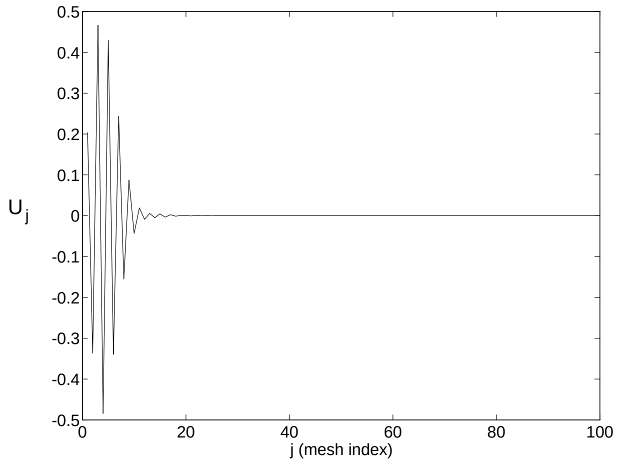

Figure 12: Right singular vector of $\mathbf{C}^{12}$ for the unstable scheme, $M = 100$.

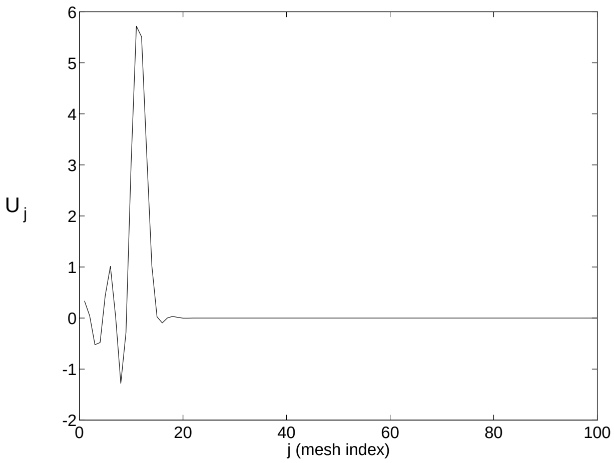

Figure 13: Right singular vector of $\mathbf{C}^{12}$ for the unstable scheme after 12 time steps, $M = 100$.

Fig. 14 shows the behavior of $\|\mathbf{C}^n\|$ if no NBS is used at the inflow boundary. With the influence of the inflow NBS removed, the instability becomes clear. With each increase in M, the maximum value of $\|\mathbf{C}^n\|$ increases. The right singular vector associated with the maximum singular value of $\mathbf{C}^n$ for $M = 100$ and $n = 76$ is shown in Fig. 15. It is a wave packet with a wavenumber corresponding closely to that producing the maximum instability according to Fourier analysis. Thus the wave packet grows in amplitude as it convects across the domain. After 76 time steps, the solution shown by the dashed line in Fig. 15 results. The growth is very slow, as expected for such a mild instability, producing a norm of 1.0779 after 76 steps. If the value of M had been chosen sufficiently large, the instability of the interior scheme would have revealed itself even with the inflow NBS included. However, the required value of M would be close to 2000. Numerical problems can occur when computing the L_2 norm of such a large matrix raised to a large power.

Conclusions

We have examined a number of issues relating to the practical use and reliability of stability tests applicable to higher-order finite-difference methods. Criticisms of the asymptotic stability test when applied to non-normal matrices have been addressed, leading to the following conjecture: the combination of stability by Fourier analysis and asymptotic stability is sufficient for Lax-Richtmyer stability and is necessary and sufficient for P-stability. In addition, an example has been given for which the pseudospectrum fails to reveal a mild instability. Finally, we have also shown that the singular value decomposition can be a helpful tool in stability analysis.

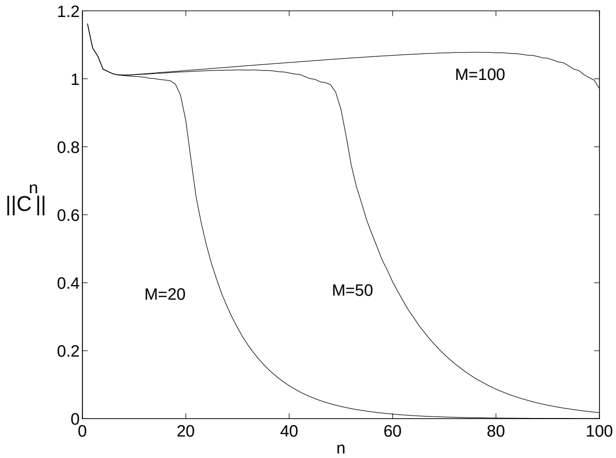

Figure 14: $\|\mathbf{C}^n\|$ for the unstable scheme without the inflow NBS.

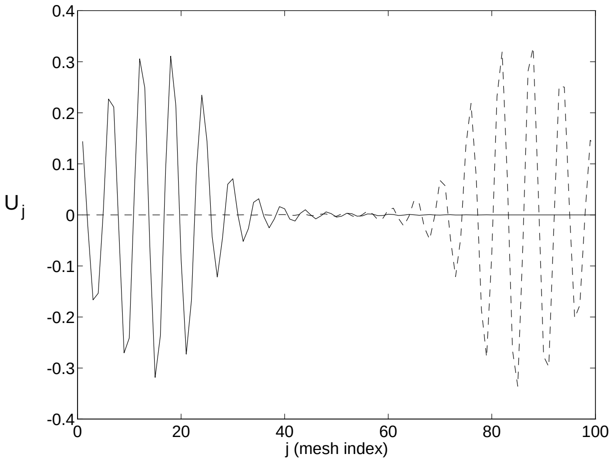

Figure 15: Right singular vector of $\mathbf{C}^{76}$ for the unstable scheme without the inflow NBS (solid line). The dashed line shows the result after 76 time steps.

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Appendix

This Appendix gives the entries of the matrix $\mathbf{A}$ for the unstable scheme. The fully-discrete matrix is found from:

$$\begin{aligned}\mathbf{C} &= \mathbf{I} + C\mathbf{A} + (C\mathbf{A})^2/2 + (C\mathbf{A})^3/6 \\ &+ (C\mathbf{A})^4/24 + (C\mathbf{A})^4/120 + (C\mathbf{A})^6/720\end{aligned}$$

The nonzero entries in the first row of $\mathbf{A}$ are (to four decimal places): 1.9833, -4.2500, 4.0000, -2.5833, 0.9500, -0.1500. The nonzero entries in the second row are: 1.1000, -1.1667, 1.0000, -1.2500, 0.5667, -0.1000. These two rows correspond to the NBS given in eqs. 13 and 14. The nonzero entries in the third row are: -0.1904, 0.8305, -0.1000, -0.6695, 0.1096, -0.0068, while those in the fourth row are: 0.0266, -0.1904, 0.8305, −0.1000, −0.6695, 0.1096, −0.0068, corresponding to the interior operator. The nonzero entries in the final three rows, which correspond to fifth-order operators, are: 0.0333, −0.2309, 0.9319, −0.2353, −0.5681, 0.0691 for the third-last row, −0.0691, 0.4477, −1.2668, 2.3131, −1.2712, −0.1537 for the second-last row, and 0.1537, −0.9914, 2.7536, −4.3414, 4.6190, −2.1935 for the last row.