Convergence of turbulence statistics: random error of central moments computed from correlated data

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Abstract

A new formula is derived for the random error of sample central moments from correlated data which does not assume an underlying distribution and is accurate to leading order in the number of sample elements. Central moments, being important quantities in turbulence research, require accurate error estimation. Many approaches have been followed in the past for estimating the random errors of central moments from correlated data. These include: simple extensions of the formula for independent data, using the formula for the random error of generic averages, assuming an underlying normal distribution, and using block bootstraps. All of these approaches are compared with the present formula using datasets from a turbulent boundary layer, freestream grid turbulence, and a turbulent round jet. For even-order sample central moments, many of the existing approaches perform well with differences of less than 15%. However, for odd-order sample central moments, only the block bootstrap methodology performs similarly well. For the same sample central moments, the other methods differ by as much as 200%–1000%.

Keywords: turbulence, error, central moments, correlated, convergence

1 Introduction

Without some quantification of the error associated with a measurement, the measured values are meaningless. The aim of error analysis is to estimate the error associated with a measurement where the error includes both bias error and random error, which determine the accuracy and precision of the measurement, respectively. There are many sources which discuss error analysis in the context of fluid dynamics (e.g., Moffat, 1988; Abernethy et al, 1985; Coleman and Steele, 1989). In particular, these describe the various sources of error that contribute to the total error associated with a measurement and how to combine them to arrive at a total error (or uncertainty) in the measurement. Contributions to the error include precision and calibration of the measuring instrument, the applicability of an equation used to compute a quantity, and the randomness inherent in an experiment due to stability of the measuring instrument, the environment, and the quantity itself (for example turbulent flows are naturally random and so will always have a large random error). In addition to assigning a level of accuracy and precision on a final result, error analysis is also useful in the planning stages of an experiment to determine the largest sources of errors such that they can be reduced or corrected for before measurements are taken (e.g., Coleman et al, 1991). In a turbulent flow, when calculating a mean, for example, the majority of the error can be random error due to the use of a finite sampling time. In order to reduce that error it is useful to accurately estimate it and determine its dependence on the sampling time. This enables the scientist or engineer to determine the sampling time necessary and plan experiments or simulations accordingly.

This paper focuses on the bias and random errors (also known as standard errors) associated with statistics that can be computed from a sample of measurements, where the individual elements within a sample cannot be

considered independent. In particular, this paper seeks to determine an accurate formula for the standard errors of central moments from correlated data, with a focus on the field of turbulence, since central moments are ubiquitous in turbulence research. In particular, for turbulent flows, measurements of the central moments are important for characterizing the turbulence: second-order moments (e.g. variance) describe the amplitude of the turbulence, third-order moments (e.g. skewness) describe the ratio of positive to negative fluctuations, and fourth-order moments (e.g. kurtosis) describe the prevalence of extreme turbulent events, for example. The skewness has also been shown to be a measure of amplitude-weighted phase in triadic scale interactions (Duvvuri and McKeon, 2015). Velocity structure functions, which involve moments of longitudinal velocity fluctuation difference (Sreenivasan and Antonia, 1997; Lavoie et al, 2005) and velocity derivative skewness (Burattini et al, 2008), which involves second- and third-order moments of the spatial derivative of velocity fluctuation, are also important statistics that describe the dynamics of turbulence. Additionally, in order to solve the Reynolds-averaged Navier-Stokes equations, one must determine the components of the Reynolds stress tensor—which involve univariate and bivariate central moments (normal and shear stresses, respectively)—by either directly modelling the Reynolds stress tensor, or by introducing additional equations to solve for the Reynolds stress tensor. The latter involve central moments one order higher, which again must be either modelled or solved with additional equations involving yet higher-order central moments. This is also true for other Reynolds-averaged transport equations. For proper modelling of these central moments, experimental measurements of them are extremely useful. Consequently, knowing the accuracy of the measurements is also of utmost importance.

The organization of this paper is as follows: Section 2 discusses the calculation of the standard error of generic averages and central moments. Section 3 presents a new formula for estimating the standard error of central moments from correlated data. Section 4 describes the experimental datasets on which the new formula will be tested. Section 5 discusses issues and best practices associated with the estimation of the standard error. Finally, section 6 provides a comparison of the formula in this paper with previous methods for estimating the standard error of central moments from correlated data.

2 Background

2.1 Standard error

We are interested in determining the uncertainty of some statistic estimated from sampled turbulent data, which is inherently random. Following Bendat and Piersol (1966), the uncertainty of a sample estimator $\bar{\theta}$ is defined as its root-mean-squared (RMS) error:

$$\text{RMS error} = \sqrt{\left\langle \left(\bar{\theta} - \mu_\theta\right)^2 \right\rangle}, \tag{1}$$

where $\langle\cdot\rangle$ denotes the expected value, which can be considered as an ensemble average over an infinite number of realizations, and μ_θ is the true value of the statistic that is being estimated, commonly referred to as the population parameter. This can be split into two terms:

$$\text{RMS error} = \sqrt{\left\langle \left(\bar{\theta} - \langle\bar{\theta}\rangle\right)^2 \right\rangle + \left(\langle\bar{\theta}\rangle - \mu_\theta\right)^2}. \tag{2}$$

The first term in (2) describes the variance of the estimator about its expected value:

$$\sigma_{\bar{\theta}}^2 = \left\langle \left(\bar{\theta} - \langle\bar{\theta}\rangle\right)^2 \right\rangle = \langle\bar{\theta}^2\rangle - \langle\bar{\theta}\rangle^2 . \tag{3}$$

This is the square of the standard deviation, $\sigma_{\bar{\theta}}$, of the distribution of $\bar{\theta}$'s, which is also referred to as the standard error.

The second term in (2) defines the square of the bias error, $b_{\bar{\theta}}$, between the expected value of the estimator and the true value:

$$b_{\bar{\theta}} = \langle\bar{\theta}\rangle - \mu_\theta. \tag{4}$$

In general, $b_{\bar{\theta}}$ is not zero; a well-known example where this is the case is the sample variance computed as $s^2 = \sum_{i=1}^{N}(x_i - \bar{x})^2/N$.

Thus, the RMS error is the root sum of squares of the standard error and the bias error of the estimator:

$$\text{RMS error} = \sqrt{\sigma_{\bar{\theta}}^2 + b_{\bar{\theta}}^2}, \tag{5}$$

and determining the uncertainty associated with the sample estimator amounts to determining the standard error and the bias error of the estimator, or equivalently, the variance and expected value of the estimator.

In the following analysis, let us assume that we have collected a sample $\{U_1, ..., U_N\}$ of some quantity, U. For example, this could be a time series of streamwise velocity in a turbulent flow, in which case we have the sample $\{U(t_1), ..., U(t_N)\}$. It does not matter what quantity is measured nor what measurement technique is used to sample the data, and that data can be sampled temporally or spatially. Furthermore, the sample can be from an experimental or numerical dataset. Some of the analysis does implicitly assume

that the data is uniformly sampled (i.e., a fixed sampling rate or distance between points) in order to make simplifications regarding sums of correlation coefficients. However, if averages are computed as numerical integrals then the same set of steps can be followed with a weighted average where the weights can hold information about the quadrature as well as non-uniform sampling. Before we begin, let us first note that the true mean of U is its expectation value (i.e., $\mu_U = \langle U \rangle$). Additionally, the true variance of U is given by $\sigma_U^2 = \langle U^2 \rangle - \langle U \rangle^2$.

2.2 Standard error of an average

Suppose we wish to compute the average over the elements in our sample and want to know the error associated with that average. In the following, it does not matter if we are computing the average $\bar{U}$ or first apply some element-wise operation on the sample before computing the average $\overline{f(U)}$, so long as there is no additional error introduced by the function f. The formulas for $f(U)$ can be obtained by substituting $f(U)$ for U in the formulas below.

Recall that we need both the expected value and the variance for the RMS error. For an average, the expected value can be computed easily:

$$\langle \bar{U} \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} U_i \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \langle U_i \rangle = \langle U \rangle \,, \tag{6}$$

where $\mu_U = \langle U \rangle$ is the true mean of U. Thus, the average is an unbiased estimator of the mean. We can also compute the variance using (3):

$$\begin{aligned} \sigma_{\bar{U}}^2 &= \langle \bar{U}^2 \rangle - \langle \bar{U} \rangle^2 \\ &= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle U_i U_j \rangle - \langle U \rangle^2 \end{aligned} \tag{7}$$

We can rewrite $\langle U_i U_j \rangle$ using the Pearson correlation coefficient:

$$\begin{aligned} \rho_{U_i, U_j} &= \frac{\langle (U_i - \langle U_i \rangle)(U_j - \langle U_j \rangle) \rangle}{\sigma_{U_i} \sigma_{U_j}} \\ &= \frac{\langle U_i U_j \rangle - \langle U \rangle^2}{\sigma_U^2}, \end{aligned} \tag{8}$$

giving

$$\langle U_i U_j \rangle = \langle U \rangle^2 + \sigma_U^2 \rho_{U_i, U_j}. \tag{9}$$

Inserting this into (7) gives

$$\sigma_{\bar{U}}^2 = \frac{\sigma_U^2}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{U_i, U_j}. \tag{10}$$

2.2.1 Independent sample elements

If the elements of the sample are all independent of one another then,

$$\rho_{U_i, U_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}. \tag{11}$$

Inserting this into (10) gives

$$\sigma_{\bar{U}}^2 = \frac{\sigma_U^2}{N}. \tag{12}$$

This is the square of the well-known equation for the standard error of the mean that can be found in most introductory statistics textbooks.

2.2.2 Correlated sample elements

Turbulent data are typically sampled at a high enough rate to resolve the smallest-scale flow features. However, due to the wide range of scales in turbulent flows, with such a high sampling rate, there is often a high degree of

correlation between adjacent elements in the sample. So, in general, $\rho_{U_i,U_j} \neq 0$, except for elements that are well-separated. If we let $k = j - i$ be the lag between the ith and jth sample elements and note that $\langle U_i U_{i+k} \rangle$ is the same for all i at the same k (i.e., $\langle U_1 U_3 \rangle = \langle U_2 U_4 \rangle = \langle U_3 U_5 \rangle$, etc.), then we have (see e.g. Bendat and Piersol, 1966; Smith et al, 2018, among others):

$$\sigma_{\bar{U}}^2 = \frac{\sigma_U^2}{N} \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \rho_{U,U}(k), \tag{13}$$

where $\rho_{U,U}(k)$ is defined the same as in (8), but with $k = j - i$ implied. The $1 - |k|/N$ factor results from there being $N - |k| = N(1 - |k|/N)$ identical terms at each k. Equivalent versions of this can be found in Liepmann (1952), Lumley and Panofsky (1964), Tennekes and Lumley (1972), and George et al (1978). Comparing (13) with (12), we see that there is an effective number of independent elements in the correlated-element case given by:

$$N_{\text{ind}} = \frac{N}{\sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \rho_{U,U}(k)}, \tag{14}$$

such that

$$\sigma_{\bar{U}}^2 = \frac{\sigma_U^2}{N_{\text{ind}}}. \tag{15}$$

All equations to this point have involved integer lags. We can convert to temporal variables, as in a time series, if we multiply (13) by $\Delta t^2/\Delta t^2$, make use of the fact that the time lag is $|\tau_k| = |k|\Delta t$, the sampling time is $T = N\Delta t$, and the autocorrelation is symmetric:

$$\sigma_{\bar{U}}^2 = \frac{\sigma_U^2}{T} \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|\tau_k|}{T}\right) \rho_{U,U}(k)\Delta t. \tag{16}$$

As $T \to \infty$, $N \to \infty$ as well for a fixed Δt. Since $\rho_{U,U}(k)$ is maximum at $\tau_k = 0$ and then drops quickly to zero as $|\tau_k|$ increases, for the range of values that $|\tau_k|$ is non-zero, $|\tau_k| \ll T$ and therefore (16) becomes:

$$\sigma_{\bar{U}}^2 \approx \frac{\sigma_U^2}{T} \sum_{k=-(N-1)}^{N-1} \rho_{U,U}(k)\Delta t \quad \text{for large } T, N. \tag{17}$$

The summation is a discrete approximation to twice the integral time scale,

$$2\mathcal{T}_{U,U} = \int_{-\infty}^{\infty} \rho_{U,U}(\tau)\,\mathrm{d}\tau. \tag{18}$$

Thus, for large N and T, we have:

$$\sigma_{\bar{U}}^2 \approx \frac{2\sigma_U^2 \mathcal{T}_{U,U}}{T}, \tag{19}$$

and the effective number of independent elements is:

$$N_{\text{ind}} \approx \frac{T}{2\mathcal{T}_{U,U}}, \tag{20}$$

This is the basis for the rule-of-thumb that there is one independent sample element for every two integral scales.

2.3 Standard errors of central moments

Many of the statistics of interest in turbulent flows can be considered central moments, where the mean is subtracted before raising to some integer power. Examples include the normal Reynolds stresses and the non-normalized skewness and kurtosis. Before we continue, we must distinguish between the true (population) central moments and the central moments computed from a

sample. The true rth central moment involves only expected values:

$$\mu_r = \langle (U - \langle U \rangle)^r \rangle \,. \tag{21}$$

Meanwhile, the rth sample central moment involves only averages:

$$\overline{u^r} = \overline{\left(U - \bar{U}\right)^r} = \frac{1}{N} \sum_{i=1}^{N} \left(U_i - \bar{U}\right)^r \,. \tag{22}$$

Since $\bar{U}$ has its own error that must be propagated, the formulas for the standard errors in the preceding sections cannot be used for central moments, which sets them apart from moments about the origin (true: $\mu'_r = \langle U^r \rangle$, and sample: $\overline{U^r}$).

2.3.1 Independent sample elements

Starting from the definition in (3), similar to what was done for an average in the sections above, Kendall (1943) derived a formula for the error variance of sample central moments assuming independence between sample elements that is accurate to order N^{-1}:

$$\sigma^2_{\overline{u^r}} = \frac{1}{N} \left(\mu_{2r} - \mu_r^2 - 2r\mu_{r-1}\mu_{r+1} + r^2 \mu_2 \mu_{r-1}^2 \right) . \tag{23}$$

This formula was introduced to the fluids community by Benedict and Gould (1996).

2.3.2 Correlated sample elements

Unfortunately, there is no formula in the literature, to the authors' knowledge, for the true standard error of central moments computed from samples with correlated elements, even to leading order.

The conventional assumption is that we can apply (23) if the sample elements are correlated by replacing the N in that expression by an effective number of independent samples, in analogy with (15). For example, Benedict and Gould (1996) give the following recommendation in their presentation of (23): "If the sampling rate is too high to ensure independence of samples, N should be adjusted to reflect the number of integral scales in the total sampling time, not the actual number of samples." Mathematically, this implies:

$$\sigma^2_{\overline{u^r}} = \frac{1}{N_{\text{ind}}} \left(\mu_{2r} - \mu_r^2 - 2r\mu_{r-1}\mu_{r+1} + r^2 \mu_2 \mu_{r-1}^2 \right) \tag{24}$$

However, there is some lack of clarity in the literature about how N_{ind} should be calculated. In particular, should autocorrelations (and integral scales) of U or u^r be used in (14) and (20)? Regardless, there is no guarantee that this approach is valid for all moment orders.

Simple formulas for the standard error of central moments can also be obtained if some assumptions about the data are made. Lenschow et al (1993) provides formulas for the second through fourth central moments if the data are sampled over a time much longer than the integral time scale and follow a normal distribution. Assuming normality and an exponentially-decaying autocorrelation, Lenschow et al (1993) also derived formulas for the first four central moments that are valid for all T; approximations to these are also provided if the sampling time is much longer than the integral time scale. Additionally, approximate formulas are derived in Lenschow et al (1993) if some non-normality is introduced to a Gaussian process. While these formulas are likely suitable for some turbulent flows, there are plenty of others for which the data is neither normally-distributed nor has a strictly exponentially-decaying

correlation. A potentially more accessible version of Lenschow et al (1993) is Lenschow et al (1994).

For second-order central moments, Wyngaard (1973) applied the formula for an average in (13) based on the work of Lumley and Panofsky (1964). By doing so, they consider the random error in the variance of fluctuating quantities, but ignore the effect of the random error contained in the sample mean. This is equivalent to using (13), but with U replaced by $f(U) = (U - \langle U \rangle)^2$. Sreenivasan et al (1978) then considered the extension of this approach to arbitrary moment order. As we will show, this approach is actually not an issue for the second-order central moment, but becomes problematic for higher-order, and more so for odd-order, central moments.

Salesky et al (2012) also devised a clever filtering method that uses a fitting routine to determine the error variance at a specific sampling time without the need for computing integral time scales. However, since this method is based on Lumley and Panofsky (1964), it also neglects the effect of the error associated with the sample mean and so is not suitable for central moments of order greater than $r = 2$.

We will assess these approaches in section 6 on three different turbulent flows.

2.4 Resampling methods

Other methods have also been devised to estimate the standard error of arbitrary statistics. These techniques, called resampling methods, include the bootstrap (Efron, 1979) and the jackknife (Tukey, 1958). The idea behind resampling techniques is to create many new samples from the original sample, to estimate the sampling distribution of the estimator from those (re)samples, and then extract the variance of the estimator from the distribution. In their

original forms, these resampling methods assume independent and identically distributed random variables. For correlated data, this can be problematic if the resampling breaks up the correlation structure in the original sample. To overcome the requirement of independence, one could attempt to downsample the sample to ensure independence between elements. This is possible so long as the standard error is associated with only one integral scale and that integral scale is known.

Fortunately, there are a host of resampling methods—primarily block bootstraps—that have been developed for dealing with correlated data (Lahiri, 2003). One form of the block bootstrap that estimates the standard error of estimators for correlated data with a higher degree of accuracy is the moving block bootstrap (MBB, Künsch, 1989; Liu and Singh, 1992). Garcia et al (2006) and Richards et al (2018) have shown its usefulness in estimating standard errors in the context of turbulent fluids. For any bootstrap method, there is some additional parameter tuning that is required. In particular, a number of replications needs to be determined in order to ensure convergence of the calculated standard error. Additionally, for block bootstraps, a specific block length needs to be determined which must be chosen large enough such that it retains the correlation of the initial signal (i.e. it requires an estimate of the integral scale), although automatic optimal block selection algorithms exist in the literature (e.g., Politis and White, 2004).

Since block bootstraps can be relatively complicated to implement, require some tuning for optimal performance, and can be computationally expensive, a simple and accurate formula for the standard error of central moments is sought for correlated data. Furthermore, bootstrap techniques should be validated against an expression for the true error variance or standard error, if such a thing exists for the quantity under investigation, to ensure that they converge to the correct value.

3 Formula for standard error of central moments from correlated data accurate to order N^{-1}

Starting from the definition of the error variance in (3), we derive an expression for the true error variance of sample central moments of arbitrary order where there is correlation between sample elements and where the underlying data has an arbitrary distribution. This approach follows the steps of Kendall (1943) in their derivation of (23), but removes the assumption of independence between sample elements. The full derivation of the expected value and error variance is contained in Supplementary section S.1.

Since the complete expression is unwieldy and impractical to implement, especially for large moment order, we present a formula for the error variance that is accurate to leading order in the number of sample elements (order N^{-1}), as in Kendall (1943). This results in a formula for the standard error that is accurate to order $N^{-1/2}$. For the bias error to the same order, we require the expected value to order $N^{-1/2}$ as well. To these orders:

$$\langle \overline{u^r} \rangle = \mu_r, \tag{25}$$

and,

$$\begin{aligned} \sigma^2_{\overline{u^r}} = \frac{1}{N} \Bigg[& (\mu_{2r} - \mu_r^2) \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \frac{\rho_{u^r,\, u^r}(k)}{\rho_{u^r,\, u^r}(0)} \\ & -2r\mu_{r-1}\mu_{r+1} \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \frac{\rho_{u,\, u^r}(k)}{\rho_{u,\, u^r}(0)} \\ & +r^2\mu_2\mu_{r-1}^2 \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \frac{\rho_{u,\, u}(k)}{\rho_{u,\, u}(0)} \Bigg]. \end{aligned} \tag{26}$$

Comparing (26) to (23), we see that for correlated data each term in (23) must be multiplied by a different correlation sum. Equivalently, this means that each term has a different effective number of independent sample-elements. If we generalize the definition of the effective number of independent sample-elements to (c.f. (14)):

$$N_{\text{ind}}(X,Y) = \frac{N}{\sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \frac{\rho_{X,Y}(k)}{\rho_{X,Y}(0)}}, \tag{27}$$

then (26) can be rewritten as:

$$\sigma^2_{\overline{u^r}} = \frac{\mu_{2r} - \mu_r^2}{N_{\text{ind}}(u^r, u^r)} - \frac{2r\mu_{r-1}\mu_{r+1}}{N_{\text{ind}}(u, u^r)} + \frac{r^2\mu_2\mu_{r-1}^2}{N_{\text{ind}}(u,u)}. \tag{28}$$

Since $\rho(0) = 1$ for autocorrelations, scaling by $\rho(0)$ in (27) is equivalent to (14) when $X = Y = U$. Comparing (28) with (24), we see that the conventional assumption that N can be replaced with a single N_{ind} in (23) could lead to a poor approximation.

If the true mean, $\langle U \rangle$, were known and used in place of $\bar{U}$ in (22), $\sigma^2_{\overline{u^r}}$ would only consist of the first term of (26). In fact, since $\langle U \rangle$ does not add any error, this would be equivalent to (13) with $f(U) = (U - \langle U \rangle)^r$, which is the approach of Sreenivasan et al (1978). The first term in (26) is related to the error involved in taking a sample average of the central moment, the third term is related to the error of the sample mean, and the second term is related to the interaction of these two.

It can be shown that for $r = 1$, the error variance in (26) is exactly zero since $\mu_1 = 0$ by construction. Since the second and third term in (26) also involve μ_1 when $r = 2$, they are zero and it can be shown that the error variance of $\overline{u^2}$ is calculated in the same way as is conventionally assumed in (24)

so long as N_{ind} is calculated using correlations (or integral scales) of u^2. Incidentally, this means that the approach of Wyngaard (1973) and Sreenivasan et al (1978) for $r = 2$ is also correct. For higher-order moments ($r > 2$) though, the error variance in (26) differs from the conventional assumption in (24) and the approach of Sreenivasan et al (1978).

For a time series, we can convert to temporal variables by multiplying the numerator and denominator of (26) by Δt^2. As (26) is only true for large N, and consequently large T, relative to the correlation sums, (26) can be written as (c.f. (16)–(20)):

$$\sigma^2_{\overline{u^r}} \approx \frac{2}{T}\left[\left(\mu_{2r} - \mu_r^2\right)\mathcal{T}_{u^r,\,u^r} - 2r\mu_{r-1}\mu_{r+1}\mathcal{T}_{u,\,u^r} + r^2\mu_2\mu_{r-1}^2\mathcal{T}_{u,\,u}\right], \tag{29}$$

where we use the zero-lag-scaled ρ in the definition of the integral time scales, and further assume that the sampling rate is high enough (or Δt small enough) that the correlation sums closely approximate the integral time scales:

$$2\mathcal{T}_{X,Y} = \int_{-\infty}^{\infty} \frac{\rho_{X,Y}(\tau)}{\rho_{X,Y}(0)}\mathrm{d}\tau \approx \sum_{k=-(N-1)}^{N-1} \frac{\rho_{X,Y}(k)}{\rho_{X,Y}(0)}\Delta t. \tag{30}$$

This implies that,

$$N_{\text{ind}}(X,Y) \approx \frac{T}{2\mathcal{T}_{X,Y}}. \tag{31}$$

The extension to bivariate central moments (e.g. $\overline{uv}$) is straightforward, although more algebraically intensive, and the result is contained in Supplementary section S.2.

4 Experimental datasets

To assess the different methods in section 2.3.2 relative to the more accurate formula presented in section 3, we consider hot-wire data of the streamwise

velocity from three experimental datasets: (1) a turbulent boundary layer (TBL), (2) freestream grid turbulence (FST), and (3) a turbulent round jet (TRJ).

For the TBL data, hot-wire measurements were made in a zero-pressure-gradient TBL at a momentum thickness Reynolds number of $Re_\theta = \theta U_\infty/\nu = 3000$, where θ is the momentum thickness, U_∞ is the freestream velocity, and ν is the kinematic viscosity. The TBL was developed over a 0.5-inch aluminum plate that extended the width and length of the 5 m-long test section of a recirculating wind tunnel with a cross-section measuring 0.8 m-high $\times$ 1.2 m-wide. The hot wires were operated at an overheat of 1.6 using a Dantec StreamLine Pro 90N10 constant temperature anemometer. The hot-wire data were filtered at 10 kHz before being sampled at 25 kHz for $T = 180$ s using a National Instruments PCIe-6259.

The FST data is case D from Dogan et al (2016) and Dogan et al (2017). It was saved as 15 samples lasting 25 s each at a sampling rate of 20.5 kHz. Thus, to construct correlation coefficients, an ensemble average was performed over the 15 samples, with the mean and standard deviation being computed over the entire 15-sample dataset. For this data, the freestream turbulence intensity was $u_{\text{rms}}/U_\infty = 12.2\%$, where u_{rms} is the root mean squared freestream fluctuations, and the Taylor microscale Reynolds number was $Re = u_{\text{rms}}\lambda/\nu = 630$, where $\lambda = u_{\text{rms}}\sqrt{15\nu/\epsilon}$ is the Taylor microscale and ϵ is the turbulence energy dissipation rate.

The TRJ data is from Sadeghi and Pollard (2012). The measurements were taken at a location $15D$ downstream of the nozzle exit along the centerline, where $D = 73.6$ mm is the diameter of the nozzle exit, with a jet velocity of $U_j = 10.4$ m/s. This corresponds to a Reynolds number of $Re_D = DU_j/\nu = 5 \times 10^4$. The data were sampled at a rate of 30 kHz with a sampling time of

$T = 120$ s. More details of the experimental setup can be found in Sadeghi and Pollard (2012).

5 A note on estimation

All of the formulas for the standard errors that we have introduced in sections 2 and 3 above involve expected values—true (or population) means, moments, and correlation coefficients—which we typically do not know. In order to estimate the standard error from a sample we have to use sample averages, moments, and correlation coefficients. For example, (29) becomes:

$$s^2_{\overline{u^r}} = \frac{2}{T}\left[\left(\overline{u^{2r}} - \left(\overline{u^r}\right)^2\right)\hat{\mathcal{T}}_{u^r,\,u^r} - 2r\,\overline{u^{r-1}}\,\overline{u^{r+1}}\hat{\mathcal{T}}_{u,\,u^r} + r^2\,\overline{u^2}\left(\overline{u^{r-1}}\right)^2\hat{\mathcal{T}}_{u,\,u}\right], \tag{32}$$

where the $\hat{\mathcal{T}}$s are estimates of the integral time scales. There is inherent error associated with replacing the true central moments with sample central moments. However, from (29), we also know that the variance of central moments goes as N^{-1}. This means that the error in replacing the true central moments with sample central moments will go as N^{-2}, which we have neglected.

The estimates of the integral scales are a bit more complicated. With finite data, as in a sample, calculation of the correlation coefficient at the largest lags has significant error due to the fact that there are fewer correlations available to average over (for lag k, there are $N-|k|$ correlations). There are two versions of the sample correlation coefficient that are typically used to estimate the

true correlation coefficients:

$$\hat{\rho}_{\text{biased}}(k) = \frac{\frac{1}{N}\sum_{i=1}^{N-|k|}(X_i - \bar{X})(Y_{i+k} - \bar{Y})}{s_X s_Y} \tag{33}$$

$$\hat{\rho}_{\text{unbiased}}(k) = \frac{\frac{1}{N-|k|}\sum_{i=1}^{N-|k|}(X_i - \bar{X})(Y_{i+k} - \bar{Y})}{s_X s_Y}, \tag{34}$$

where $\bar{X}$ and $\bar{Y}$ are the sample means and s_X and s_Y are the sample standard deviations of X and Y, respectively. Equation (33) is called the "biased" correlation coefficient because when $\bar{X}$ and $\bar{Y}$ are replaced by the true population means, the expected value of $\hat{\rho}$ is biased relative to the true ρ, while $\hat{\rho}$ from (34) is not. These labels are a bit of a misnomer for the estimators as written since it is possible for the "biased" estimator to have less bias error than the "unbiased" estimator when sample means are used (see e.g., Percival, 1993). Additionally, due to the lack of available data at the largest lags, the "unbiased" estimator has a lot of random error at those k. Conversely, since $\hat{\rho}_{\text{biased}}(k) = (1-|k|/N)\hat{\rho}_{\text{unbiased}}$ those same large lags are attenuated and $\hat{\rho}_{\text{biased}}$ better estimates the true ρ, which is typically close to zero at large k. For these reasons, the "biased" estimator is typically the better choice. As an example, the correlation for the second term in (26), and implicit in (29) and (32), would be estimated using the biased sample correlation coefficient as:

$$\hat{\rho}_{u,\,u^r}(k) = \frac{\frac{1}{N}\sum_{i=1}^{N-|k|} u_i(u^r_{i+k} - \overline{u^r})}{\sqrt{\overline{u^2}\left(\overline{u^{2r}} - (\overline{u^r})^2\right)}}, \tag{35}$$

where we have made use of the fact that $\bar{u} = 0$ by definition. For $r \geq 1$, $\overline{u^r} \neq 0$.

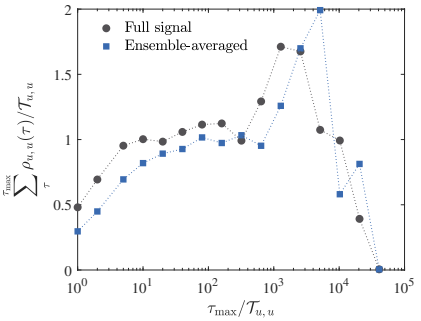

Fig. 1 Comparison of the cumulative sum of the ensemble-averaged correlation coefficient of varied record length vs the full signal correlation coefficient for time-series data of streamwise velocity at $y^+ = 20$ in a TBL.

However, there are some peculiarities about the biased estimator for the correlation coefficient, as discussed in Percival (1993). In particular, the sum over all lags of (33) can be shown to be exactly zero, despite that not being the case for the true correlation coefficient. In all of the sums that we consider, there is an additional factor of $(1 - |k|/N)$, but this only has significant attenuation for large lags. These summations converge to a non-zero number, but not the sum of the true correlation coefficent. Thus, in practice, to estimate an integral scale, it is necessary to sum up to some maximum lag. Smith et al (2018) provides some guidelines for estimating integral scales. In particular, for short non-periodic signals, they suggest using the familiar first zero crossing of the correlation coefficient. For longer signals, they suggest breaking the sample into smaller records that are about 20 integral scales in length and summing the ensemble-averaged correlation coefficient to obtain an estimate of the integral scale, where parent statistics are used (i.e., the mean and variance of the full signal are used for each record). In figure 1, we show using a TBL time series at $y^+ = 20$ that the results obtained in this way for different record lengths ($\tau_{\max}$) actually mirror the results obtained by simply summing the correlation of the full signal up to $\tau_{\max}$, but require slightly longer $\tau_{\max}$ to get the same result. The reason for this is that ensemble averaging actually

reduces the statistics slightly relative to the full signal. For the full signal, $\hat{\rho}(k)$ is the sum of $N - |k|$ correlations. For ensemble averaging, if each record is of length n, then $\hat{\rho}_i(k)$, the correlation coefficient at lag k for the ith record, is the sum of $n - |k|$ correlations. If we ensemble average $\hat{\rho}_i(k)$ over the N/n records, we get a total of $N - N|k|/n$ correlations used in the computations. This says that for each additional record used for ensemble averaging, there are $|k|$ fewer correlations used; correlations that would otherwise span across the record boundaries are eliminated with the ensemble averaging.

For a more robust estimation of correlation sums (including integral scales), we suggest borrowing the method in Politis and White (2004), used to determine optimal block sizes for block bootstraps. This method essentially involves convoluting the correlation with a window function, similar to what is done in spectral estimation, and performing the sum over that. In fact, the connection between the correlation and the power spectrum is how this method was conceived (Politis and Romano, 1995). In this particular case a trapezoidal window is used; other window functions are surely possible, and may provide better estimates, but the window size (or bandwidth) would need to be modified from the present method. For simplicity, we suggest using the method exactly as described in Politis and White (2004):

1. Estimate $\sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \rho(k)$ by $\sum_{k=-M}^{M} w\left(\frac{k}{M}\right) \left(1 - \frac{|k|}{N}\right) \hat{\rho}(k)$, where ρ is the true correlation and $\hat{\rho}$ is the estimated correlation, M is the bandwidth, and $w\left(\frac{k}{M}\right)$ is the window function.
2. Choose $\hat{m}$ where $|\hat{\rho}(\hat{m} + k)| < c\sqrt{\log_{10} N/N}$ for $k = 1...K_N$, where $K_N = \max(5, \log_{10} N)$ and $c = 2$ are the suggested parameters, and select $M = 2\hat{m}$.

3. Construct the trapezoidal window:

$$w\left(\frac{k}{M}\right) = \begin{cases} 1, & \text{if } 0 < |\frac{k}{M}| < \frac{1}{2} \\ 2\left(1 - |\frac{k}{M}|\right), & \text{if } \frac{1}{2} < |\frac{k}{M}| < 1 \\ 0, & \text{otherwise} \end{cases}.$$

By virtue of the implied hypothesis test in step 2, this method chooses the bandwidth based on the lag where the correlation remains close to zero for a minimum of five lags. While this is similar to the first zero-crossing for exponentially decaying correlations (summing up to the first zero-crossing would be akin to using a rectangular window with $M = \hat{m}$), it is robust in that it also gives reasonable estimates for periodically-decaying correlations (or other correlations with multiple peaks so long as the correlation eventually decays to zero). This is useful because the limitation of estimating (29) is in obtaining accurate estimates of the integral scales. For $N \gg M$, the $1 - \frac{|k|}{N}$ factor is negligible. A comparison of the integral time scale of u computed from the TBL dataset using this method and both summation up to the first zero-crossing and up to 20 integral times scales is shown in figure 2. The trapezoidal window method produces similar results to the other two methods and seems to be a good compromise. In section 6, we use this trapezoidal window method with $\hat{m}$ and M calculated using $\hat{\rho}_{u,u}$, which typically has the longest integral time scale, to compute all integral time scales in (32). This ensures a consistent bandwidth.

6 Convergence of central moments

To quantify the accuracy of using the approaches in section 2.3.2, a percentage difference is considered relative to the more accurate formula presented in

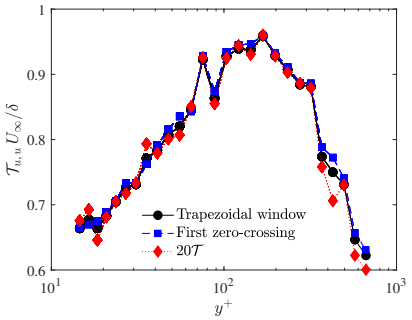

Fig. 2 Comparison of the estimated integral time scale of u for three different methods across a TBL.

section 3 (denoted $\sigma^2_{\overline{u^r},\,\text{present}}$):

$$\Delta\sigma^2_{\overline{u^r}} = \frac{\sigma^2_{\overline{u^r}} - \sigma^2_{\overline{u^r},\,\text{present}}}{\sigma^2_{\overline{u^r},\,\text{present}}} \times 100\%. \tag{36}$$

The approaches compared include: the Gaussian formulation with (denoted as Gaussian/Exp. ρ) and without (denoted as Gaussian) an exponential autocorrelation in Lenschow et al (1993), the conventional assumption of extending (23) to correlated data with two choices of autocorrelation variable (denoted as Conventional $(u,\,u)$ and Conventional $(u^r,\,u^r)$), the approach of Sreenivasan et al (1978) of extending (13) to higher moment orders (denoted as SCA78), and the moving block bootstrap (denoted as MBB).

Figures 3–5 show this percentage difference for the three flows mentioned in section 4 for moment orders from $r = 2$ to $r = 7$. For the TBL this data was extracted at $y^+ = 75$; located in the logarithmic region, this is where the data most closely approximates a Gaussian distribution. Similarly, the TRJ data was extracted at $r/r_{0.5} = 0$ (i.e. the jet centerline) for the same reason. For the TBL, TRJ, and FST data, we have that $T/\mathcal{T}_{u,\,u} \approx 45\,000$, $T/\mathcal{T}_{u,\,u} \approx 20\,000$, and $T/\mathcal{T}_{u,\,u} \approx 11\,000$, respectively. Thus, $T \gg \mathcal{T}$ for all of these flows ($\mathcal{T}_{u,\,u}$ is

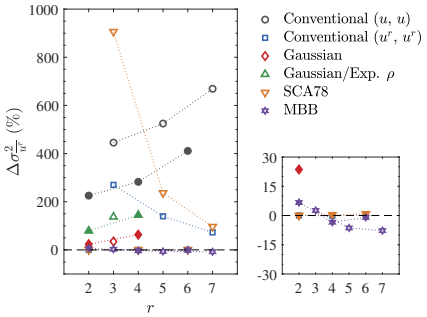

Fig. 3 Difference in the error variance for the first seven central moments of U computed from the methods in section 2.3.2 relative to (29) for the TBL data at $y^+ = 55$. The dotted lines track the results for the even (filled symbols) and odd (open symbols) moments separately. Inset expands the region near $\Delta\sigma^2 = 0$.

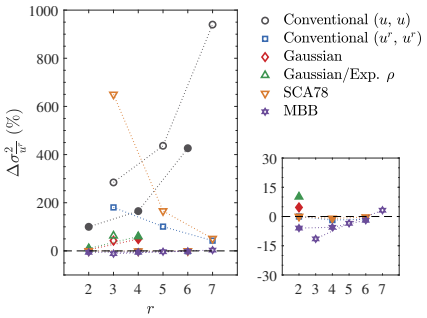

Fig. 4 Difference in the error variance for the first seven central moments of U computed from the methods in section 2.3.2 relative to (29) for the TRJ data at $r/r_{0.5} = 0$. The dotted lines track the results for the even (filled symbols) and odd (open symbols) moments separately. Inset expands the region near $\Delta\sigma^2 = 0$.

the longest integral time scale), as required for (29) and many of the methods mentioned in section 2.3.2.

The trends for each of the flows are very similar. Typically, the difference is smaller for even moments than it is for odd moments. In particular, at $r = 2$, as was mentioned in section 5, Conventional (u^r, u^r) is exactly equivalent to (29), as is SCA78. For higher-order even moments, both of these methods agree with (29) to within 5% for all of the flows considered. For $r = 3$, SCA78 has a very large discrepancy for all flows, being off by $> 600\%$ for all flows considered, but tends towards 0% as odd r increases. Conventional (u^r, u^r) behaves similarly

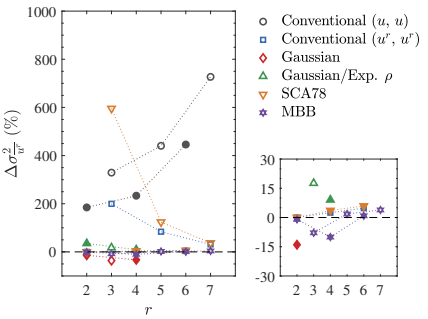

Fig. 5 Difference in the error variance for the first seven central moments of U computed from the methods in section 2.3.2 relative to (29) for the FST data. The dotted lines track the results for the even (filled symbols) and odd (open symbols) moments separately. Inset expands the region near $\Delta\sigma^2 = 0$.

for odd r, but has a smaller discrepancy at $r = 3$ of $\sim 200\%$. If Conventional (u, u) is used then there is a very large discrepancy for all r, but more so for odd r, that increases with r. The Gaussian method shows a lot of dependency on the flow, and this is not unexpected since the accuracy of the method is likely tied to how closely the flow resembles a Gaussian process. They are all within 30% of (29) for $r = 2$, but show greater discrepancy as r increases, being off by $\approx 50 - 100\%$ at $r = 4$. As expected, assuming an exponential form for the autocorrelation coefficient, in addition to a Gaussian distribution, typically results in larger discrepancies with (29). The MBB performs the best across all r, including odd r, being within 15% for all r and for all flows. In most cases, $\Delta\sigma^2$ is positive, meaning that the sampling time required to reach a target level of error is actually smaller than implied by the other methods in section 2.3.2.

To assess how each of the methods performs when the flow deviates even further from a Gaussian distribution and as the integral scales change, we compute the percentage difference with (29) for various wall-normal locations in the TBL data. This is done for $r = 3$ (figure 6) and $r = 4$ (figure 7). As an estimate of the degree of non-normality in the flow, the quantity $Q = \sqrt{S^2 + (3 - K)^2}$

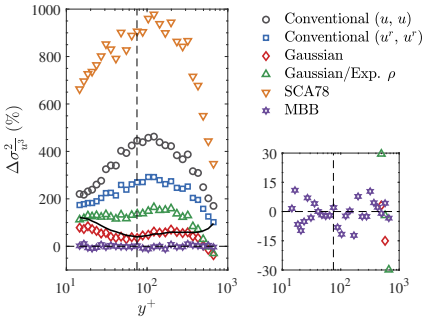

Fig. 6 Difference in the error variance for the $r = 3$ central moment of U computed from the methods in section 2.3.2 relative to (29) for the TBL data across y. The thin black line is $Q = \sqrt{S^2 + (3 - K)^2}$ in arbitrary units. Vertical line corresponds to minimum of Q at $y^+ = 75$. Inset expands the region near $\Delta\sigma^2 = 0$.

is also displayed alongside the results, where $S = \overline{u^3}/(\overline{u^2})^{3/2}$ is the skewness and $K = \overline{u^4}/(\overline{u^2})^2$ is the kurtosis ($S = 0$, $K = 3$, and therefore $Q = 0$ for normally-distributed data). This curve shows good qualitative agreement with the results from the Gaussian approximations; the closer to a Gaussian the data is (i.e. Q close to zero), the closer the Gaussian approximations are to the result of (29). Again only the MBB does a good job of estimating the error variance for both even and odd order moments, although SCA78 and Conventional (u^r, u^r) perform similarly well for $r = 4$ across the boundary layer. In figure 6, the results from Conventional (u^r, u^r), Conventional (u, u), and SCA78 seem to follow the same trend as $\mathcal{T}_{u,u}$ in figure 2. This is due to the fact that (29) depends on $\mathcal{T}_{u,u}$, whereas Conventional (u^r, u^r) and SCA78 only depend on $\mathcal{T}_{u^r,u^r}$. Conventional (u, u) also follows this trend because it uses $\mathcal{T}_{u,u}$ for terms that should depend on $\mathcal{T}_{u^r,u^r}$ based on (29).

As a final point, for the MBB, its accuracy is dependent on the number of bootstrap replications, n_{rep}, used to estimate the error variance. Figure 8 shows the convergence of the MBB standard error relative to (29) for the $r = 3$ central moment at $y^+ = 75$ using the TBL data. Due to the random nature of the MBB, each time it is performed, a separate convergence history

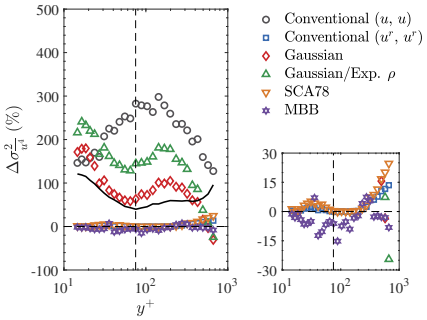

Fig. 7 Difference in the error variance for the $r = 4$ central moment of U computed from the methods in section 2.3.2 relative to (29) for the TBL data across y. The thin black line is $Q = \sqrt{S^2 + (3 - K)^2}$ in arbitrary units. Vertical line corresponds to minimum of Q at $y^+ = 75$. Inset expands the region near $\Delta\sigma^2 = 0$.

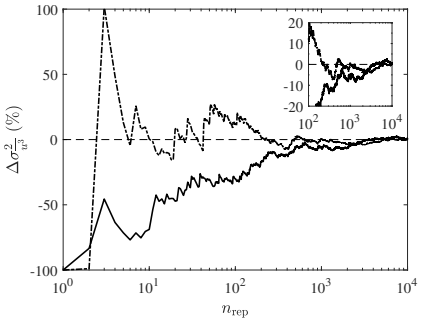

Fig. 8 Convergence of the difference in the error variance for the $r = 3$ central moment of U computed from the MBB relative to (29) for the TBL data at $y^+ = 75$. Inset expands the region near $\Delta\sigma^2 = 0$.

is generated, as is seen for the two plots in figure 8. For an accuracy of $< 5\%$, the MBB requires $n_{\text{rep}} > 1000$ for this central moment. For figures 3–5, we have used 1000 replications in the computation of the MBB estimate of $\sigma^2_{u^r}$ as a compromise between accuracy and run time. Additionally, to properly estimate the error variance using the MBB, it is imperative to construct the replications from the initial U data, and not data with the mean subtracted. This ensures that the sample mean, $\bar{U}$, remains a random variable. Not doing so would produce results akin to SCA78.

7 Summary and conclusions

We have derived a formula for estimating the standard errors of sample central moments of arbitrary order from correlated data which does not assume an underlying distribution and is accurate to leading order in the number of sample elements. Using experimental data of streamwise velocity in a turbulent boundary layer, freestream grid turbulence, and a turbulent round jet, we show that there are sizable discrepancies between methods that have previously been used to estimate the variance of central moments and the present formula. For even-order central moments, good agreement ($< 15\%$ difference) is seen when either the conventional assumption using autocorrelations of u^r, the method of Sreenivasan et al (1978), or the moving block bootstrap (MBB) is used for all flows. In particular, when $r = 2$, the former two methods are exactly equivalent to the formula derived in the present work. For odd-order central moments, however, good agreement is seen for only the MBB.

To ensure accurate estimates of the error variance or standard error of central moments, it is imperative that (26) or the temporal version in (29) be used. Previous approaches have either ignored the effect that the sample mean has on the standard error of sample central moments or made incorrect assumptions about how to extend the existing formula for independent sample-elements. Block bootstrap methodologies, such as the MBB, can also be used so long as the sample mean is not removed from the data before forming the bootstrap replications and a sufficient number of replications are used ($n_{\mathrm{rep}} \gtrsim 1000$).

Supplementary information. Derivations of the expressions for the expected value and variance of the sample central moments, as well as additional details, are included as supplementary information.

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Declarations

Ethical approval. Not applicable.

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

Authors' contributions. R.B. developed the theoretical framework, carried out the TBL experiments, processed the experimental data, drafted the first and final versions of the manuscript, and prepared the figures. D.W.Z. and P.L. provided supervision, acquired funding, and edited the manuscript. P.L. assisted in the acquisition of the FST and TRJ datasets.

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Availability of data and materials. The TBL data used in this study are available upon reasonable request from the corresponding author.

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